EXPERIMENTS WITH THE DISPLACEMENT INTERFEROMETER

By CARL BARUS
Hazard Professor of Physics and Dean of the Graduate Department in Brown University

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PREFACE.

The present volume contains applications of the displacement interferometer to subjects which suggested themselves from time to time. Unfortunately it was not possible, in the laboratory of Brown University, which is situated on a hill in the middle of a large city, to carry out any experiment to its final degree of rigor. Quiet surroundings, a location free from tremor, and irregular temperature variations would have been necessary. But the development of methods of the kind in question was nevertheless quite feasible; and without attempting to push them to a limit, the range of application could be fully investigated.

Among the subjects selected for treatment was the horizontal pendulum. In the first part of Chapter I certain available forms of the pendulum, with and without a float, are considered and tested as to their discrepancies, through long lapses of time, by a reflection method. Among the interesting results obtained is the suggestion of an apparatus capable of measuring changes of elongation to the amount of even less than 4×10^{-10} of the total length per vanishing interference ring.

In the second part of the chapter the interferometer itself is used, a serviceable method of application worked out, and the range of application studied through many months. With a relatively very wide scope (several seconds of arc) there should be no difficulty, under proper surroundings, of measuring changes of inclination as small as 3×10^{-4} seconds of arc per interference ring, and it is probable that one could reach smaller angles by modifying parts of the pendulum.

In Chapter II an attempt is made to use this interferential horizontal pendulum for the measurement of the gravitational attraction of two parallel disks. What was obtained, however, was a definite repulsion of the disks, decreasing with their distance apart and appreciable even within 1.5 mm. of this distance. As the method of measurement contemplates the viscosity of the film of air between the disks, and as the effect of any natural charge or potential would be insignificant in comparison with the forces observed, it is probable that the repulsion in question is attributable to the molecular atmospheres by which the disks are surrounded in air, supposing that such atmospheres of gas increase in density as the surface of the disk is approached.

Chapter III is introduced as a severe test on the interference equation employed for the case of path differences resulting when glass columns as much as 10 inches long are inserted in one of the component beams of the displacement interferometer. It appears that the constants of any dispersion formula may be obtained directly from these observations. The equations for the relations of displacement and wave-length increments show, however, that the anticipation of great precision in the determination of refraction, by lengthening the column of glass, is not fulfilled. The ellipses become proportionately more sluggish in their motion as the path difference is increased.

In Chapter IV a number of incidental experiments, on allied subjects, have been grouped together. In the first paper the possible bearing of certain disk colors of circular gratings on the somewhat similar phenomenon in coronas is discussed. The second paper deals with the performance of the easily available film grating to replace the ruled-glass grating, for purposes of displacement interferometry, from a practical standpoint. With the same end in view the third paper considers the use of the Nernst filament as an available illuminator, in the absence of the arc lamp or sunlight. In conclusion, an interesting case of regular reflection and refraction of *scattered* light, bearing on the X-ray phenomena recently discovered by Professor Bragg, is treated in the fourth paper.

In Chapter V, finally, following the suggestive experiments made in an earlier report, the displacement interferometer is directly applied to the quadrant electrometer. In the several hundreds of adjustments made no serious difficulty was encountered in the optical parts of the experiments, and that was the question chiefly at issue. The sensitiveness obtained in this way should have been of the order of a millionth of a volt per vanishing interference ring; but owing to the uninterrupted commotions surrounding the laboratory already referred to, possibly also to difficulties residing in the electrometer, this limit could not be reached. The experiments, therefore, largely explore the available scope of the method.

My thanks are due to Mrs. D. T. Knight and to Miss R. R. Snow for efficient assistance in connection with the preparation of the papers for the press.

CARL BARUS.

BROWN UNIVERSITY, *September 25, 1915.*

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CHAPTER I.

ELLIPTIC INTERFEROMETRY APPLIED TO THE HORIZONTAL PENDULUM.

PART I.—DIRECT DIFFERENTIAL REFLECTION.

1. Introduction. Method.—Before using interferences, it seemed interesting to apply the interferometer adjustment to the case of simple reflection, the mutual displacement of the two direct images from the front and rear face of the mirror on the pendulum being used for the measurement of the angle of deviation of the pendulum. In such reflection from a glass plate there is necessarily considerable loss of light; but at radii of 20 and 30 meters, when the source of light is a slit closely in front of a Nernst filament, this difficulty is not prohibitive. It is necessary, however, that the lens of the collimator, as well as the plates of glass and mirror used, be of high optical quality; otherwise it is impossible to obtain sharp condensation at a very distant focus. One may also concentrate the slit images to a point by a cylindrical lens, placed with its linear element at right angles to the direction of the slit.

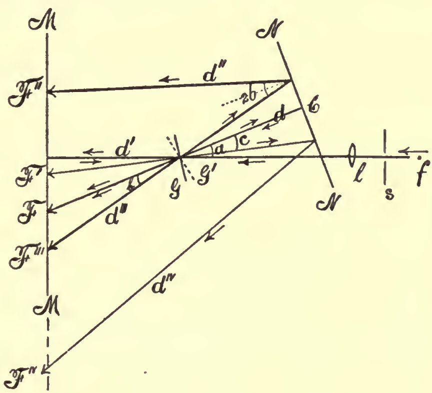


Fig. 1.

The interesting feature of the method is that it is *independent* of any zero-point, as the distance apart of the two images on the far screen at once measures the inclination of the pendulum axis, the normal position being that in which the two images coincide. If the two opaque mirrors are rigidly fixed, the direct or incident beam of light from the source and the subsidiary reflecting mirrors may shift without modifying the datum for the inclination. Furthermore, the sensitiveness is twice that of the case of single reflection, other things being equal. The method is thus particularly adapted for the measurement of the inclination of the plumb-bob relatively to the earth.

The annexed diagram, fig. 1, will make the method clear. Here s is the fine slit in front of the Nernst filament f, and l the condensing doublet (about 60 cm. focal distance; for rough work an ordinary spectacle-glass answers very well) of the collimator. It is necessary that this lens be wide, weak, and good in order that a sharp focus and as little loss of light as practicable be obtained at the distant focus.

The pencil from the collimator strikes the plate of glass G at the end of the horizontal pendulum, the greater part of this, d', being transmitted to the opaque mirror M, the remainder, d, reflected from the opaque mirror N. It is advantageous to have M and N equidistant from G, as nearly as practicable (20 or 30 cm.), and far from the lamp f, to avoid the menace of temperature discrepancies. If the mounting is of gas-pipe, water circulation might be used, but this is not necessary.

In the diagram the pencils d and d' are normally reflected at M and N. On returning d is transmitted and d' reflected, so that the beams reunite and proceed together to the far focus F, 20 or 30 meters distant, where they are caught on a paper or ground-glass screen, or directly observed with the lens. It is particularly necessary that the movable reflector at G be an excellent optical plate, 1 or more inches square. When the plate at G (which is at the extremity of the horizontal pendulum) rotates over a small angle θ, the reflected rays d'' and d''' now diverge in opposed directions from the center C on the face of the opaque mirror N, and pass to the distant foci F'' and F''', where they are now at a distance x apart. If the rotation of the horizontal pendulum were $-\theta$, the positions of the beams would be exchanged (see F' and F^{iv}). In other words, if the pendulum vibrates, the two foci F'' and F''' move in opposed directions, passing through each other, when the normal position is instantaneously assumed, irrespective of the *amplitude* of vibration. It may be noted that a similar adjustment may often with advantage be attached to any ordinary pendulum.

The mounting of the plate G and the mirrors M, N, etc., is identical with that of the interferometer described in the next section (the plate G is there replaced by a transparent grating) and need not be treated here. Necessarily the collimator and the mirrors M and N are attached to the same pier which carries the horizontal pendulum, to the end of which the mirror G is attached; but the horizontal pendulum and its case must otherwise be quite independent of the goniometric apparatus.

2. Equations.—It will be convenient to suppose the foci F, F'', F''' to lie in the plane of the mirror M, and the two mirrors M and N to be equidistant, so that $d=d'$, taken from the plate G as an ideal plane. Let a be the angle between the normal to N and the direction of the incident pencil; *i.e.*, let a be the angle between the mirrors, made as small and as convenient as practicable. Then if the mirror at G rotates over any angle $\theta=b/2$, the distance apart x of the foci F'' and F''' will be

$$(1)\qquad x=d\frac{\sin 2b}{\cos(a-b)}\left(\frac{1}{\cos a+b}+\frac{1}{\cos b}\right)$$

This equation may be transformed to

$$(2)\qquad x=4d\tan b\left(\frac{\cos^2 b}{\cos 2a+\cos 2b}+\frac{1}{2}\,\frac{\cos b}{\cos(a-b)}\right)$$

Since the angles a and b are invariably small, the cosines may be expanded, so that

$$(3) \qquad x = 2d \tan b \left(\frac{1-b^2}{1-(a^2+b^2)} + \frac{2-b^2}{2-(a-b)^2} \right)$$

is nearly true, or

$$(4) \qquad x = 4d \tan b \left(1 + \frac{3a^2}{4} - \frac{ab}{2} \right)$$

Since $b = 2\theta$ is exceedingly small (but a few seconds),

$$(5) \qquad x = 8d\theta \left(1 + \frac{3a^2}{4} \right)$$

Finally, if a is also sufficiently small, which will usually be the case, and D is $2d$ or $2d'$, so that the distances to the far screen, F'' and F''', are measured from the mirror N,

$$(6) \qquad x = 4D\theta$$

One may note in passing that the distance over which the N ray travels from coincidence is

$$x_1 = \frac{d \sin b}{\cos (a-b)} \left(2 + \frac{1}{\cos a} \right)$$

whereas the M ray travels over

$$x_2 = d' \left(\tan (a+b) - \tan a \right)$$

where $x = x_1 + x_2$. Hence, for small angles, the N ray travels over 3 times the distance db of the M ray, the total being $4db$. Thus the angle of deviation θ is measured by x, apart from any other consideration, except that the distance D is very large and therefore invariable and the sensitiveness is twice as large as in the case of single reflection.

To test this result in its practical aspects, a millimeter micrometer was installed at the end of the pendulum, at a distance of 51.5 cm. from the axis. The two images traveled in opposite directions, in steps, from end to end of a 30 cm. scale, while the micrometer was moved forward 1 mm., successively, eight times, as follows:

Micrometer.	Mean.	x	Micrometer.	Mean.	x
0.0 cm.	16.1 cm.	32.3 cm.	0.5 cm.	14.5 cm.	− 4.9 cm.
.1	15.7	24.7	.6	14.2	−12.3
.2	15.4	17.3	.7	13.9	−19.7
.3	15.1	9.9	.8	13.7	−27.0
.4	14.8	+2.5			

With the exception of the first and last readings the steps of x are 7.4 cm. and equidistant. Hence

$$\theta = .1/51.5 = 7.4/4D$$

whence D is equal to 914 cm., which agreed with the direct measurement. The center of the images ("mean") holds pretty well to the scale, shifting but from 13.7 to 16.1, while the distance D is smaller and the total angle (0.8/51.5 radian, about 1 degree) larger than would usually be employed.

In the experiments made below, the distance D was frequently above 2,000 cm. Since 1 second of arc is about 5×10^{-6} radians, the deflection x corresponding to $\theta=1$ sec. would therefore be

$$x=4\times2000\times5\times10^{-6}=4\times10^{-2}\text{ cm.}$$

or nearly half a millimeter. A sharp focus F'', F''' is thus nevertheless needed if single seconds of θ are to be read off visually. I frequently made use of what seemed to be the internal diffraction patterns of the slit, fine bright lines in each being used for measurement.

The angle θ, denoting the deviation of the pendulum, is invariably very large as compared with the angle α, the corresponding change of inclination of the pier to the plumb-line. In fact, fig. 2, cdg denotes the horizontal pendulum, with the grating at g, pivots at c and d, the center of gravity at G, at a distance h from the axis cd. The latter prolonged intersects the plumb-line through G at e, all in the plane of the diagram. The angle between the axis de and the vertical df is φ in the same plane. When the axis, owing to the tilt of the pier, takes a new position de', the arc ee' is nearly

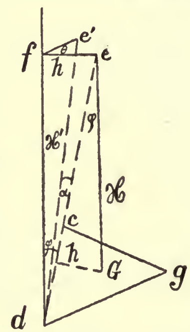


FIG. 2.

$$h'\theta=H'\alpha$$

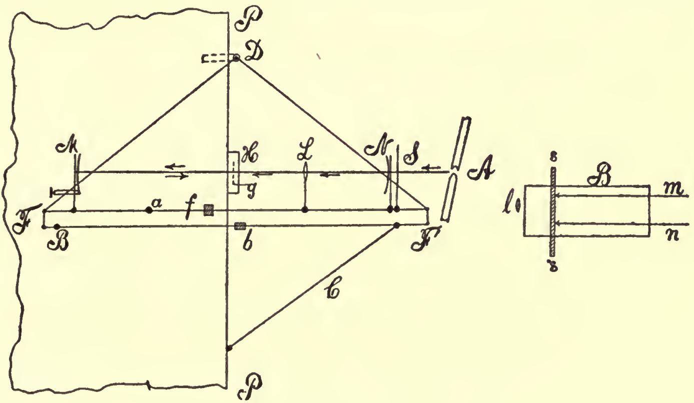
When φ is very small, as is necessarily the case, $h=h'$ and $H=H'$ very nearly, so that

$$h\theta=H\alpha \text{ and } h=H\varphi \quad \text{whence } \alpha=\varphi\theta$$

Thus if de is a rigid stick pivoted at d and fe a flexible inextensible line, the motion is such as if the whole mass of the pendulum were concentrated at e, the diagram being the plane of the couple $Mgh=MgH\varphi$.

As in $\alpha=\varphi\theta$ all angles are given in radians, if the angle φ is of the order of 1°, the ratio α/θ is but 0.0175. I need merely instance, therefore, if $\varphi=0.01$, since $X=4D\alpha/\varphi$ and $D=2{,}000$ cm. (as above, conditions all of which are easily realized), that per second of arc of α, $x=4$ cm.

The form which the scaffolding eventually took is shown in the diagram, fig. 3, in elevation. All rods were of ¼-inch iron pipe; so that, if desirable, a current of water could have been passed through the essential braces. FF' is a long rectangle of gas-pipe, 240 cm. from end to end and 10 cm. high. Its direct attachment to the pier *PP* is at *b* and *f*. The ends of *FF'* are braced by the rods *FD* and *F'D* in a vertical plane, and by rods at *B* (horizontal) and *C* (oblique). The lower abutment of *C* is about a meter down toward the rear, so that *D*, *F'*, *C*, *B* is a large tetrahedron. This arrangement is at the same time adequately simple and firm, but it is not of course proof against tremors. In fact, not a method was found by which these could be excluded entirely. They exist in the pier. The optical parts are now attached to either of the horizontal rods *FF'* by strong clamps of the usual type (reëntrant wedges). From the lamp at *A*, which may be either an arc or a Nernst fila-



FIGS. 3 and 4.

ment, the light passes successively through the micrometer slit *S*, the collimating lens *L*, to the vertical plate of glass *H* on the horizontal pendulum. Thence it is reflected to the opaque mirror *N* about 20 cm. behind the diagram, and transmitted to the opposite mirror *M*. From both mirrors it is returned to the plate of glass at *H*, after which the nearly coincident reflected and transmitted beams pass to the left of the diagram to the far distant screen (2,000 cm.) on which they are caught and their distance apart, *x*, measured.

The method best suited for visual observation, which alone is here attempted, consisted in adjusting a clear glass millimeter scale (fig. 4) *ss*, about 15 cm. long, seen distinctly through the lens *l*, and noting the position of the arriving beams of light, *m* and *n*, practically in focus on *ss*. A dark box open at both ends *B* surrounds the beams and *l* is moved on a slide.

The opaque mirrors (here plane) are necessarily adjustable around horizontal and vertical axes and the micrometer slit must be very fine. At *a* is a fine adjustment (horizontal and vertical axes) for the mirror *M*, though it need only be used in the interferometry below. The gas-pipes, when partly screwed, partly clamped, together, make a very serviceable framework for experimental purposes.

For convenience in observation, it is necessary that the horizontal pendulum be damped. A water damper, as well as an oil damper, was used without apparent disadvantage, care being taken to keep the long copper trough in which the vane dips full of water. Soldered parts must be covered with a varnish, for instance of wax and resin, as there will otherwise be slow galvanic corrosion and a precipitate.

3. Observations.—The observations were carried out for experimental purposes only, because an adequate lens and plate-glass mirrors were not at hand, and because the hill on which the laboratory is built is in a continual state of tremor, due to the heavy car freightage, both on the surface of the hill and through it. The pier, moreover, was not protected and insulated to an extent needed in refined seismological work. Hence the chief purpose of these experiments is to indicate the deviations to be expected prior to the interferometer work of the next section. In the preliminary experiments, the pendulum was used without a damper and changes of inclination of the horizontal pendulum of $\alpha=0.4$ second per day, or even 1 second in several successive days, were not infrequent. On other days the pendulum was relatively fixed. The succession of points was quite regular, and maxima and minima frequent.

Observations of a more definite character were taken between September 26 and December 3, 1913. They were computed throughout and charted. They, however, have little more than the local interest specified, and I will therefore merely give an example of part of the results in the graph, fig. 5. By equation (6) above, where x is the distance apart of the two images of the slit and D the intervening space between the plane of these images and the fixed mirror N nearest the lamp,

$$\theta=x/4D,$$

θ being the angular deflection of the pendulum. Since $D=2{,}000$ cm.,

$$\theta=125\times10^{-6}x \text{ radians.}$$

Furthermore, $\alpha=\varphi\theta$, where φ is the inclination of the line drawn through the points of the pivots to the vertical, and α the change of inclination of the pier to the vertical corresponding to θ. Hence,

$$\alpha=\varphi x/4D$$

The value of φ found below in the work with the interferometer is $\varphi=0.0108$ radian, so that α is a little larger than 1 per cent of θ. Thus

$$\alpha=1\cdot35\times10^{-6}x \text{ radians } = \cdot28x \text{ second, nearly.}$$

In fig. 5, the values of α in seconds are given from November 4 to November 26. The apparatus was interfered with from time to time, as shown at a, and new modifications were introduced. The water damper was used throughout, so that the pendulum did not vibrate.

The curve as a whole represents the contortions of the pier, probably while it was being gradually dried out from its moist condition of the summer by the steam heat radiating from the steam-pipes in the room. This was particularly apparent toward the end of the curve, at a time when the room for incidental reasons happened to be excessively hot. But a continual increase of α is apparent, showing that the structure as a whole was gradually tipping in one direction. Blasting operations were in progress in a tunnel underneath the hill, but it is not probable, judging from later results, that these affect

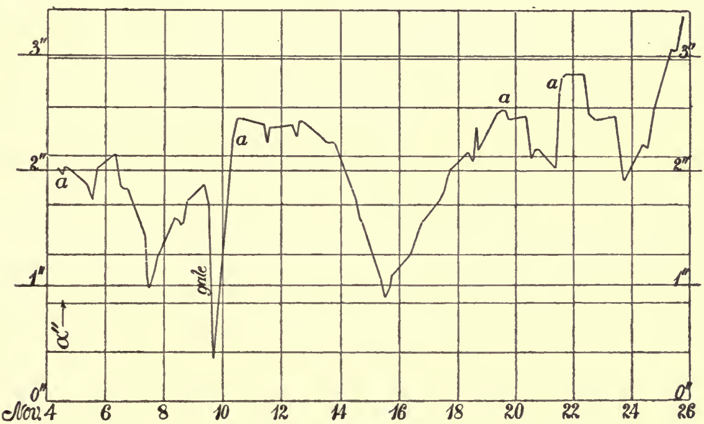


FIG. 5.

the readings of the horizontal pendulum in the lapse of time. It is impossible to come to definite conclusions at present, but it is not out of the question that actual seasonal changes in the hill itself have also been recorded. Thus the occurrence of a gale always produces a marked temporary effect. I have not, however, thought it worth while to compare the graph of fig. 5 with other graphs (temperature, etc.) which were simultaneously taken, as such work, to be trustworthy, must be done in this laboratory in the summer months, and it will then be advantageous to use the interferometer, as shown below.

4. New apparatus, without float.—To mount the symmetrical form of pendulum described below, an iron scaffolding was installed (in the absence of a suitable pier), erected on the cement foundation layer of the physical laboratory. The truss sustaining the optical parts is shown in perspective in fig. 6 and the horizontal pendulum independent of this, *i.e.*, free from it, in fig. 7. In practice the apparatus, fig. 6, surrounded fig. 7. The feet *BA*, *BC*, *B'A'*, *B'C'* of the framework are bolted to the firm layer of cement at *AC* and *A'C'* and carry the horizontal rods *GH*, which with *DI* make a parallelogram. *FGE* is the rod for mounting the optical parts, secured by the braces *DG* and *DEcaFD* in a vertical plane. Lateral braces *EH*, *EI*, *FH*, *FI*, *A'J*, *C'K* provide for firmness in the horizontal direction. The feet *ABC* and *A'B'C'* were additionally cross-braced (not shown). This truss is not ideal; but as it had to pass through a square hole in the floor of the room,

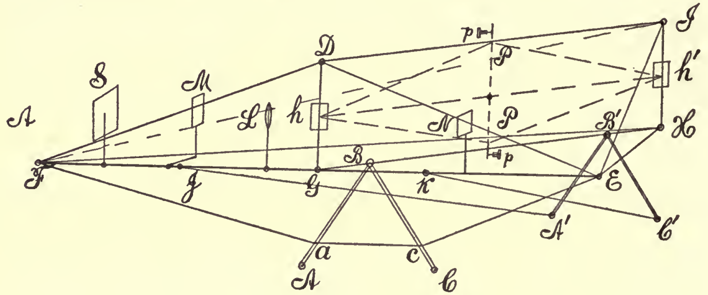


FIG. 6.

50 cm. above the cement floor, conditions had to be compromised. It seemed, however, to meet the initial requirements of the experiment adequately. All rods are of ¼-inch gas-pipe, usually screwed fast at one end and clamped at

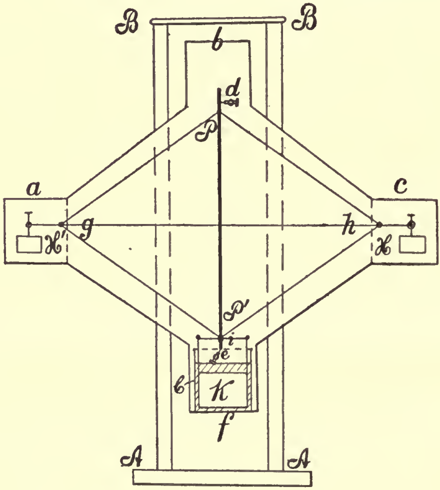


FIG. 7.

the other. The position of the horizontal pendulum on a separate mounting is sketched in at *hPh'P'*, the glass plate being at *h* and *h'*, the pivots at *p* and *p'*.

The two adjustable opaque mirrors are shown at *M* and *N*, the former being about 20 cm. to the rear, and held by clamps. *S* is the slit and *L* the lens of the collimator, the lamp being at A, on a separate stand. In fig. 7, the tall standard $AABB$ of 1-inch brass pipe, well braced (not shown), supports both the case *abcf* of tin plate and the pivot supports *de* of the horizontal pendulum $HPH'P'$. The rear and sides of the case are rigidly fixed, but the front may be removed as a whole. Similarly, the square boxes *abcf*, of which *a* and *c* are provided with glass plates, slide out horizontally or vertically, both in front and in the rear. The pendulum is thus easily accessible for adjustment. The pivots may be revolved around a horizontal axis, or moved fore and aft, right and left, or up and down. The fore-and-aft movement is provided with a screw adjustment, like the right-and-left movement. The method of attachment is much the same as that to be described below.

Pendulum and case are quite independent of the truss (fig. 6), a very *essential* condition, as the truss must often be touched for optical adjustment. In a later adjustment the case also was mounted in complete independence of the pendulum.

The present symmetrical horizontal pendulum was made of ¼-inch aluminum tubing, the vertical brace PP of ⅜-inch aluminum tubing. The junctions are brass tubing and the cups and slots for pivots either jeweled or of glass-hard steel. There is a slot for the *d* pivot and a conical hollow for the *e* pivot. Inasmuch as the horizontal pendulum is invariably under tremor, with the consequent absence of static friction, the pivot *e* in the first experiment was adjusted vertically, though the means were at hand for adjusting it in any inclined direction, as will presently be shown. The horizontal brace *gh* is of tense brass wire, forming a rhombus when seen from above, the object being to enhance the lateral rigidity of the pendulum. Finally, the plane parallel glass plates, H, H', lie to the front of the pendulum, so that a beam of light may pass across, parallel to its plane, as called for in some of the interferometer measurements. They are below the line of horizontal symmetry and each is adjustable around a horizontal and a vertical axis.

The case at *f* was specially adapted for the installation of a float K, which will be described presently and was not used in the first experiments. The vat C of the float must be supported on an entirely separate standard (not shown).

5. Observations.—The total weight of the apparatus, including the mirrors, was $M = 740$ grams, with the center of gravity at $h = 13.9$ cm. from the axis; the effective length of the arm was $R = 59.5$ cm. and the period $T = 29$ seconds.

The moment of inertia for an axis through the center of gravity was found to be 1.51×10^6. Since the mass was 740 grams, this is equivalent to a radius of gyration $i_0 = 45.2$ cm. Hence, since the distance of the center of gravity from the pivotal axis is $h = 13.9$ cm., the radius of gyration for the same axis will be $i = 47.3$ cm. From this and the above period,

$$\varphi = \frac{4\pi^2 i^2}{T^2 gh} = 7.71 \times 10^{-3} \text{ radians,}$$

or about 0.44 degree. Hence, the change of inclination of the pendulum is nearly $\alpha = 0.0077\theta$. The force at a distance $R = 59.5$ cm. from the pivotal axis (place of the plate of glass or grating) is

$$F_R = Mgh\varphi\Delta N/R = 11\Delta N,$$

supposing the interferometer to be used and that ΔN is the displacement of the micrometer. Thus F_R here and in the preceding case, when referred to the same φ and ΔN, shows

$$F/\varphi\Delta N = 1080 \text{ and } 4000, \text{ respectively,}$$

so that the present horizontal pendulum is about four times more sensitive than the other if the same interference method is used. This, however, was

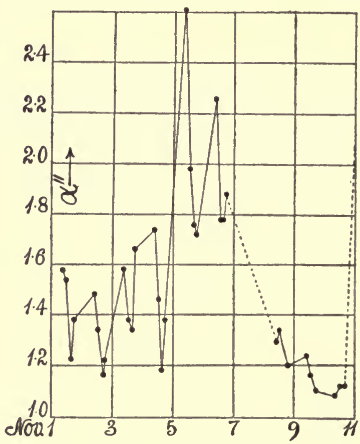


FIG. 8.

not desirable at the outset, the above method of reflection admitting of easier interpretation. Hence, $\theta = x/4D$ nearly, where x is the distance apart of the two slit images corresponding to the distance D.

The results obtained for the current changes of inclination, α, in seconds of arc, are given in fig. 8, on the same plan as the preceding. The work was continued for nearly two weeks, but only a brief example is shown in the figure.

In the present experiments the whole weight of the pendulum, 740 grams, rests directly on the lower vertical steel pivot, and the friction is correspondingly large and probably excessive. Static friction, however, was not present, the pendulum being continually in motion. On November 7 certain changes of apparatus were undertaken, and a new zero had to be selected. As a whole, the changes of inclination of this pendulum lie within 1.5 seconds of arc, and they agree in order of values with the data of the steel pendulum attached to the pier. Values of α, however, made at the same time, do not show the same run of variation, which may be reasonable, as the present pendulum is erected on the concrete subfloor of the laboratory. It was supposed that some change of inclination would result from the possible shifting of the lower pivot in its socket after the jar following an explosion, but there is no certain evidence of this. The chief purpose of these experiments is thus the comparisons which they will offer with the cases of the following paragraphs where the pendulum is partially supported on a float.

6. New apparatus, with float. Horizontal pivots.—A thin cylindrical sheet-copper float was now prepared, about 10 cm. in diameter and 10.3 cm. high, weighing with appurtenances 175 grams. It was attached symmetrically to the axis, and the method of submergence is shown in fig. 7, where K is the float and C the water-vat. Three vertical brass wires about 2 mm. in diameter, 120° in horizontal angle apart, pass from the disk i through the surface of the water. The buoyancy was found to be 627 grams by direct measurement, whereas the remainder of the pendulum weighed 697.5 grams. The effective weight was thus only 70.5 grams. Subsequently, to increase the moment of inertia and to move the center of gravity further from the axis (in the interest of greater permanence at the pivots), an additional weight was attached at the end of one arm, bringing the total weight to $M=971$ grams and the effective weight $M-V$ to 169.2 grams. The center of gravity of the solid pendulum was moved outward from 8.1 cm. to 13.1 cm. from the vertical axis. Moreover, the value of the factor is now $M/(M-V)=5.74$. Inasmuch as the greater part of the weight was supported by the float, the pivots were here tentatively placed horizontally, the lower fitting into a conical hollow of glass-hard steel with its axis horizontal; but the results shown below are quite unfavorable to this adjustment of pivot.

To find the moment of inertia with respect to an axis through the center of gravity, the pendulum was swung in its erect position, from a wire of known modulus of torsion. In this way the moment of inertia was found to be 1.610×10^6 and the minimum radius of gyration $i_0=40.71$ cm. Hence, the square of the effective radius of gyration was $i^2=i^2_0+h=1{,}830$. The period, in case of insignificant damping, was determined as $T=20$ seconds. Finally, since the pendulum is supported at the axis and not at the center of gravity, the new or flotation center of gravity is thus $h'=Mh/(M-V)$. The inclination φ of the axis, or $\varphi=4\pi^2 i^2/T^2 gh$, on inserting the values given, reduces to $\varphi=0.014$ radian, or about 0.80 degree. This is larger than necessary, but it was thought wise not to diminish it. The change of the angle of inclination, being $\alpha=\varphi\theta$, for the deflection θ, is thus determined. For a given φ it is independent of the presence or absence of the float, which therefore does not conduce to enhance the precision of the quantity α, except in diminishing the friction of the pivots.

The force at the distance $R=61.5$ cm. from the axis (this being also the position of the line of light passing through the plate) is, for like θ and φ, since

$$F_R = Mg\varphi h\Delta N/2R^2$$

on introducing the values given,

$$F_R = 133\Delta N.$$

Thus, if $\Delta N = 10^{-4}$ cm., $F_R = 10^{-3}\times 13$ dynes, or about $10^{-3}\times 4$ dynes per vanishing interference ring.

7. Observations.—The data are given for convenience in fig. 9, on the same plan as the above, the reflection method being used. Though the work was continued through several weeks, only an example is shown, as it was necessary to readjust the apparatus frequently, and the behavior was throughout anomalous. Since $\theta = x/4D$,

$$\alpha = \varphi\theta = 0.014\theta \text{ radian} = 0.81x \text{ second of arc,}$$

since $D=900$ cm., $M=971$ grams, $M-V=169$ grams, $R=61.5$ cm., $h=13.1$ cm. $T=20$ sec.

In fig. 9 the ordinate x is given in arbitrary units, which must be divided by 5.74 to reduce them to seconds of arc.

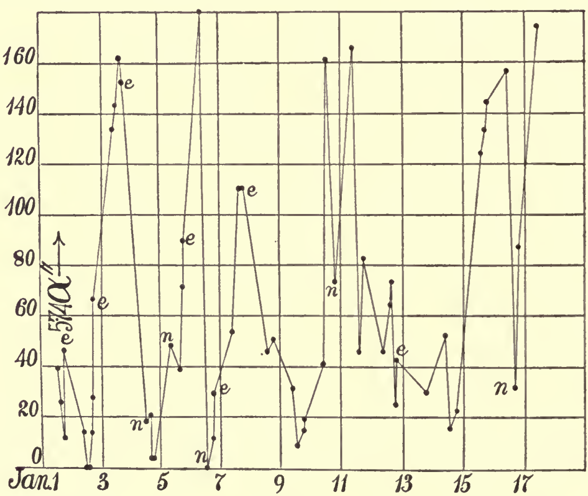


FIG. 9.

The conical sockets for the horizontal pivots of the pendulum being of steel and not quite smooth, it is possible that the relatively enormous values of the changes of inclination, α, registered may have been due to displacements of the pivots in their sockets; but readjustments of the fiducial zero (before the observations marked n in the graph) were as frequently necessary when there was no explosion (marked e in the graph) due to blasting under the hill, as after the occurrence of such a disturbance. In fact, the values of α range within 26 seconds and are often as large as that per day, whereas in paragraph 5, for the case of the apparatus without a float, the whole limit of variation was not above 1.5 seconds.

In fact, the two slit images frequently separated to a distance exceeding $x=36$ cm., or nearly as many seconds (compared with the isolated maximum of 8 cm. above), while the distance between mirror and scale was but $D=900$ cm. (compared with 2,000 cm. above). The apparent change of inclination of the line determined by the pendulum pivots is thus, in the present case of the float, registered about 17 or 18 times larger than the above similar case without a float, whereas there should be no difference. At least, it is difficult to conceive how the float, which practically compensates the weight of the pendulum, can introduce any seriously variable torque around the vertical axis.

In fig. 10a, let ABG be the horizontal projection of the lower pivot, the center of buoyancy, and the center of gravity of the pendulum. Then, with the corresponding notation, the forces involved will be $A+B=G$, the couples involved, Ah and Bh, very nearly, if h is the distance AG. The effective couple is thus Ch, the vector sum of Ah and Bh. If the angle between AG and BG is ϵ, the couple Ch may be resolved into a normal couple N and a parallel couple H equal to $Bh\epsilon$ nearly, whose axis is essentially horizontal. In fig. 10b, where AZ is the vertical and AF the line of pivots at an angle φ to AZ, the horizontal couple H may be again resolved into a couple whose axis is normal to AF, which is ineffective, and another P whose axis is parallel to AF, where $P=Bh\epsilon\varphi$, nearly. Now, although P is of the second order of small quantities, it does not follow that it is inappreciable, for if B is replaced by $V\rho g$, where ρ is the density of water at the given temperature, the buoyancy couple is $V\rho gh\epsilon\varphi$. If the observed deflection is θ, the couple due to any simultaneous change of inclination will therefore be

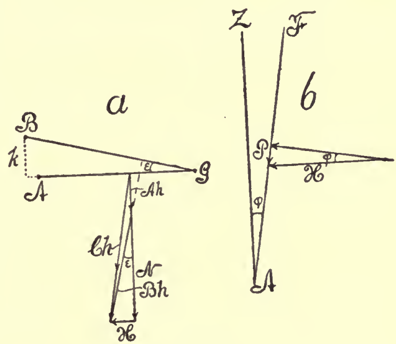


FIG. 10.

$$T=Mg\rho h\varphi\left(\theta-\epsilon\frac{V\rho}{M}\right)$$

and the tilt

$$\alpha=\varphi(\theta-V\rho\epsilon/M)$$

Hence only if ρ remains constant, will

$$\alpha'-\alpha=\varphi(\theta'-\theta)$$

with a corresponding value free from ϵ for $T'-T$.

There is another point of view from which the question may be approached: Any variation of buoyancy B, if B is eccentric, is virtually equivalent to a displacement of the center of gravity of the pendulum. This occurs when the temperature of the water, in which the float is submerged, changes. Let AG be the plane through the axis of the center of gravity of the pendulum when the float is not submerged, k the perpendicular distance of B from this plane. The center of gravity after the submergence of the float will be displaced laterally (if $V\rho$ is the mass of liquid displaced by the float)

$$k' = \frac{V\rho k}{M} = \frac{V\rho}{M} h\varepsilon$$

Since the center of gravity must lie in the same vertical plane with the line of pivots AF, the pendulum will have to rotate over an angle

$$\theta' = k'/h = V\rho\varepsilon/M$$

The observed angle θ is thus to be divided by θ' to obtain the amount due to simultaneous changes of inclination only. Of course, ϵ may be either positive or negative. Hence, the apparent change of inclination from α to α' is to be interpreted

$$\alpha' - \alpha = \varphi\left(\theta' - \theta - \frac{\varepsilon V}{M}(\rho' - \rho)\right)$$

Before discussing the question, however, it is preferable to obtain data with a more perfect pivot adjustment; in other words, to use pivots inclined toward the center of gravity and provided with jeweled bearings.

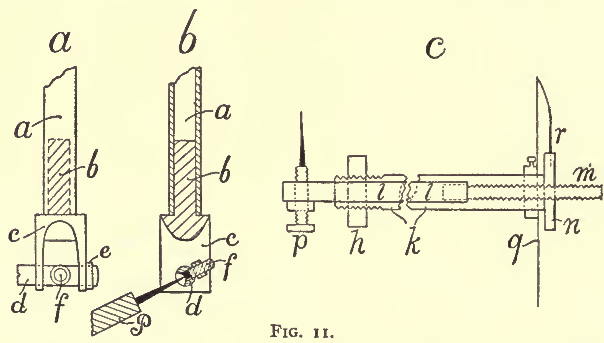


FIG. 11.

8. Second apparatus, with float. Jeweled bearings.—The anomalous results for α obtained in the last experiments were in the first place to be associated with the unsatisfactory pivots. Hence, these were readjusted so as to point toward the center of gravity of the pendulum. Moreover, the steel cup was inadequately smooth and could not be polished. It was therefore replaced by a conical hollow of polished sapphire, placed so that its axis prolonged passed through the center of gravity of the horizontal pendulum.

The manner in which the jewels were secured is shown in fig. 11*a*, and sectionally in fig. 11*b*, turned at right angles around a vertical axis to the preceding figure. Here *a* is the lower end of the central vertical tube of the horizontal pendulum into which the stem *b* of the forked holder *c* fits very snugly. The two flat prongs *c* carry the brass screw with fine thread *d*, which is horizontal and is secured in any position by the lock-nut *e*. The conically hollowed jewel, black in figure, is firmly embedded in the small brass cylinder *f*, which in turn may be screwed centrally into the wider cylinder *d* and fixed by a lock-nut. *P* is the brass stem carrying the needle of glass-hard steel which dips in the sapphire socket. The cylinders *P* and *f* are coaxial, but may be given any inclination to the vertical and then locked. As the effective weight of the pendulum does not exceed 170 grams, the strain on the pin and jewel is not excessive, and the results appear to show that they rendered excellent service. The upper pivot played in a groove of glass-hard steel as before, and it did not seem necessary to modify this.

The remainder of the horizontal pendulum was of the form already sketched in figs. 6 and 7; but precautions were subsequently taken to mount the water-bath for the float on a separate pillar, quite independent of the horizontal pendulum and its case. Later the case was also independently mounted. Trial was made of a water damper attached to the end of the beam (H' in fig. 6) on the side opposite to the mirror H. This, however, was soon discarded because of the capillary forces introduced. As a rule, the damping obtained at the float is adequate.

In order to set the zero of the pendulum at a given point, as well as to vary the inclination of the axis by the definite amount needed in the independent data of φ (see page 20), the lower pivot is virtually on a micrometer screw, capable of moving it by a definite amount z at right angles to the plane of the pendulum. This device is shown in fig. 11*c*, where *p* is the pivot screw with lock-nut securing it to the brass rod *l*. The latter fits snugly in the end of the piece of ¼-inch gas-pipe *k* and is secured by the lock-nut *h*, the pipe being longitudinally slotted within it. The rod *l* is firmly fastened to the micrometer screw *m*, the nut of which, *n*, drags the rod *l* from left to right, in spite of considerable friction, when turned clockwise. Finally the rotation of *m* is measured on a dial *q* fixed to the pipe *k*. Hence, as shown below, § 11, if a displacement z is given to the lower pivot at a distance y below the upper, $\alpha = z/y = \varphi\theta$.

9. Observations.—The constants of the pendulum were the same as in the preceding case, with the exception of the period T, which was found to be 16.2 seconds. The other constants are $M = 971$ grams; $R = 61.5$ cm.; $h = 13.1$ cm.; $M - V = 169$ grams; $i^2 = 40.7$ cm.; whence

$$\theta = 278 \times 10^{-6} x;\quad \alpha = \varphi\theta = 6 \times 10^{-6} x \text{ radians, nearly.}$$

To obtain

$$\varphi = 4\pi^2 i^2 / T^2 gh$$

the constant T only had to be changed, the former value of φ being $\varphi'=0.014$ radian and $T=20$ seconds. Hence,

$$\varphi=\varphi'\frac{400}{(16.2)^2}=0.0214 \text{ radian,}$$

or about 1.2 degrees, roughly. This is an unnecessarily large angle and is merely admitted as a first experiment, to be decreased in successive experiments. Hence, finally,

$$\alpha=\varphi\theta=6.0\times10^{-6}x \text{ radians}=1.23x \text{ seconds of arc.}$$

The apparatus is thus relatively insensitive, seeing that 1.2 seconds go to a centimeter of deflection, x.

FIG. 12.

These observations were carried on for some time in the midst of other work, during February and March of 1914, the method of reflection being used. The data are first given in fig. 12. The ordinates of the latter are in arbitrary units and must be divided by 5.74 to reduce them to seconds of arc. In the continuous record from February 23 to March 13, the range of variation is almost as large as it was in fig. 9, showing that the use of jeweled bearings has had little or no influence on the result. An interesting fact is the depression produced by the gale on March 1 (see g in fig. 12). Though the behavior of the apparatus, as such, apart from the anomalously large results, was throughout satisfactory, it was supposed that the attack of water on the soldered joints of the copper float was an objectionable feature. These were accordingly covered with resinous cement, with a removal of this trouble after March 13, the new zero being indicated at n in fig. 12; but the behavior thereafter was even more variable than before, showing that something not connected with the change of inclination is in question.

10. Observations, continued.—The horizontal pendulum was now readjusted for greater sensitiveness and for a smaller vertical inclination φ, by moving the upper pivot inward. Since T^2 varies as $1/\varphi$, considerable displacement is required to change T markedly. The period found was $T=25$ seconds, all the other constants remaining unchanged. Hence, since the original period corresponded to $T=20$ seconds, $\varphi'=0.014$ radian, the inclination is now

$$\varphi=0.014\frac{400}{625}=0.0090 \text{ radian, or } 0.51^\circ\text{, nearly.}$$

Since $\alpha=\varphi\theta$ radians, and since $\theta=x/4D$, where $D=900$ cm.,

$$\alpha=\frac{0.0090}{3{,}600}x \text{ radians}=0.515x \text{ second.}$$

Thus a centimeter of distance between the two slit images corresponds to about 0.5 second of arc of inclination of the pendulum axis.

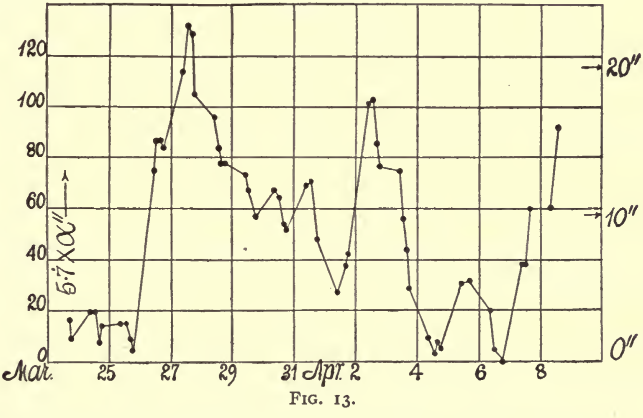


FIG. 13.

The damping was as before moderate, due only to the axial float.

The new data given in fig. 13 in arbitrary units (to be reduced 5.74 times to refer them to seconds of arc) show the same peculiarities as the preceding. The range of variation is of the usual abnormally large value. There is no adequate indication of increased sensitiveness due to larger period T which is in question. It will thus be necessary to endeavor to account for the anomalous apparent variability of α observed in these experiments with the float.

11. Effect of temperature on the float, etc.—It will be necessary at the outset to obtain the changes of buoyancy due to corresponding changes of temperature of the water in which the float is submerged. To obtain the change of buoyancy with temperature, the following table of increments may be consulted, where the normal temperature is taken as 20° and $V=802$ cm.:

$t=10°$	$V\rho_t - V\rho_{20} = +1.179$
15	$+\ .706$
20	$\pm\ .0$
25	$-\ .914$
30	-1.997

The total effect of temperature between 10° and 30° would thus be but slightly over 3 grams. This may be estimated to act upon a lever arm not exceeding 1 cm., the endeavor having been made to keep the float axial. Thus we may assume that a moment of 3×981 dyne cm. would not be exceeded in any variation of temperature, the moment being 0.15×981 per degree centigrade.

Direct preliminary experiments on the effect in terms of the deflection x of definite moments around a horizontal axis were made by placing 10 gram weights on the pan (i, fig. 7) carrying the float, at a distance of nearly 5 cm. on either side of the vertical axis of rotation. The effective moment is thus 95×981 dyne cm. The successive deflections were (differences due to deviation of pendulum during observation)

10 grams in front, $x=3.4$ cm.	10 grams in rear, $x=35.4$ cm.
6.7	33.9

This is equivalent to $x=29.6$ cm. for the given torque, or

$$x=\frac{29.6}{95\times981}=0.31/981 \text{ cm.}$$

of deflection per dyne-centimeter of torque. Hence, using the preceding estimate, the deflection should be

$$\frac{0.31}{981}\times0\cdot15\times981=0\cdot46 \text{ cm.}$$

per degree centigrade per centimeter of eccentricity, which would be equivalent to about $x=1$ cm. for a range of temperature from 10° to 30°. It does not appear, therefore, even if the eccentricity of the float is greater than was assumed, that the temperature decrement can be a menace.

The endeavors to measure the temperature discrepancy directly were all failures, inasmuch as, during the long time of cooling, the deviations of the pendulum exceeded the temperature effect, and because the necessary stirring of the water in the float interfered with the free play of the pendulum. The temperature effects obtained were as liable to be positive as negative. In fact, it is conceivable that although direct effect of temperature may not be serious, the indirect effect produced by the friction of *irregular* convection currents of water on the float may be so. Unfortunately no means of allowing for these suggests itself, so that constancy of temperature is a condition for the proper functioning of the floated pendulum. Symmetrical occurrences would of course be ineffective.

It will now be advisable to resume the equation of moments in § 7, where the torque is fully expressed as

$$T=Mgh\varphi\left(\theta-\epsilon\frac{V\rho}{M}\right)$$

if the weight mg is put on the pan of the float at a distance l from the vertical axis, $T'-T=mgl\varphi$. Consequently if ρ does not vary, the effect of ϵ vanishes, whence

$$h=\frac{ml}{M(\theta'-\theta)},$$

the distance of the center of gravity from the axis, may be found. The measurements below show this to be a good method under proper precautions.

A number of experiments of this kind were made with the pendulum modified by the addition of a damper on the left, which would throw the center of gravity slightly in that direction. The first two series of observations compared the deflection x with the water damper attached and the float either fully or less than half submerged, respectively. The variable data obtained for x and their small value showed that capillary forces were in play which completely vitiated the use of the pendulum. The zero was not steady. There seemed to be an actual capillary resistance in play. Hence the h obtained was too large. The damper at the end of the arm, notwithstanding its convenience, is therefore not admissible as an attachment under the conditions of sensitiveness of the pendulum.

With the damper removed, the float being but half submerged, the results were improved, but the capillary forces on the float were still excessive. It was not until the float was completely submerged that the capillary forces were negligible and consistent values of h were obtained, the accuracy of which might easily have been improved by closer observation.

The addition of over 500 grams weight of buoyancy has no effect on the deflection, if the error due to capillary forces is allowed for.

If the torque applied is constant while the temperature of the water in which the float is submerged changes, the differential equation becomes $d\theta/d\rho=\epsilon V/M$ or

$$dx=\epsilon 4D(V/M)d\rho=52d\rho$$

Thus the compensation would be, per degree of ϵ, at

10°	$10^5 d\rho = +147$	$10^3 dx = 76$
15	$+88$	46
20	± 0	0
25	-94	-49
30	-259	-129

Hence, for the whole range of 20 degrees, the compensation would not exceed $x=0.2$ per degree of ϵ, or about $x=1$ cm. for $\epsilon=5°$. This estimate is thus of the same order as the above, since the eccentricity of 1 cm. there postulated makes $\epsilon=1/12$ radian or 5 degrees, roughly.

Finally, inasmuch as considerable alteration had been made at the pendulum during the preceding experiments (addition of damper, etc.), it seemed advisable to redetermine φ, using, however, the micrometer method, instead of the more elaborate pendulum method. The following values of x were found for successive turns of 6° each of the micrometer screw:

Turn of screw.	x	Mean δx per 6°.
0°	29.0 cm.	18.5 cm.
6°	11.9	
12°	− 7.5	
18°	−25.7	
24°	−44.5	

The screw being a ¼-inch screw with 20 threads to the inch, its pitch may be put 0.125 cm. If z is the displacement of the lower pivot for each partial turn of 6°, y the distance apart of the pivots, and C the constant to be found,

$$\alpha = z/y = Cx \text{ radians, or, in seconds of arc, } C = 0\cdot 31$$

The value of C found for the pendulum used in the case above was $C=0.515$. The difference is larger than was expected; but with the center of gravity but 12 cm. from the axis, the addition or removal of the weights at the end of the beam 60 or 70 cm. from the axis is of marked consequence. It is also surprising that the displacement method is so consistent in its results, as nothing more than an ordinary clock-dial with a pointer was used at the micrometer. These results could easily be much improved. In other words, the present direct displacement method for $C=\varphi/4D$, and the above direct momental method for h, are not only much more simple, but on the whole more reliable than the laborious vibration methods. With M they suffice completely for the evaluation of the torque, $T=Mgh\varphi\theta$.

12. Further observations.—A final series of observations was now made with the new apparatus, recording under good conditions, in the absence of artificial heating. The trough for the float, supported entirely free from the horizontal pendulum, was provided with a thermometer, which was read off at the same time as the inclination observation. The direct measurement of φ gave $\alpha = 0.3x$ seconds of arc. These (summer) data are given in fig. 14 and the temperatures are inserted in the same figure. The work was continued for about 6 weeks, not all of the data finding room in the figure, and the graph after July 3 had to be displaced, as shown.

The new results still partake of the same tendency to enormous variations which characterize the older (winter) data. The essential error has, therefore, not been removed. On comparison with the detailed temperature curve above, however, the clue of the anomaly is obtained, for although the temperature variations are not quite contemporaneous with those of inclination, there can be no doubt of the immediate relation between them. The case is all the more

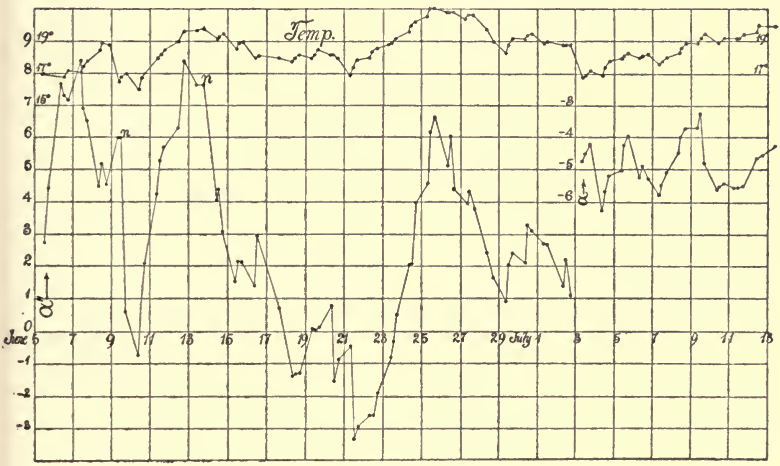


FIG. 14.

puzzling, however, as single degrees are in question, enormous changes of inclination being produced by 4 degrees. Under the circumstances, moreover, complete identity in the direction of variation of temperature and inclination graphs was not to be expected, for the temperatures given are those of the water in the float and will therefore vary more sluggishly than the temperature of the metal parts. The air temperatures, again, which were also taken, would vary faster than those of the metal, evidence for which will presently be shown. It is therefore next in order to actually examine the structure of the standard of the horizontal pendulum.

13. Effect of temperature on the scaffolding.—To give the columnar support of the horizontal pendulum adequate steadiness, it was braced from behind as shown in fig. 15. It was not foreseen that any menace could lurk in such a system, such as was later detected. In fig. 15, *ABC* is a side-view of the brass vertical standard (in duplicate, as shown in fig. 7), the horizontal pendulum being supported between A and B, while the heavy base CD rests on foot-screws on the cement subfloor of the basement of the laboratory. BD is the brace in question, extending almost half-way up the standard at an angle of about $\theta=14°$ to it. For ordinary quiet surroundings this truss seemed to be adequate, as the water-vat of the float was held on a separate standard, free from the pendulum and its case. The pendulum, in view of

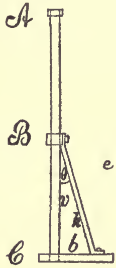


FIG. 15.

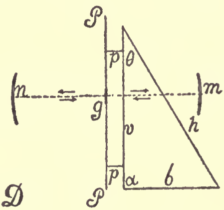


FIG. 16.

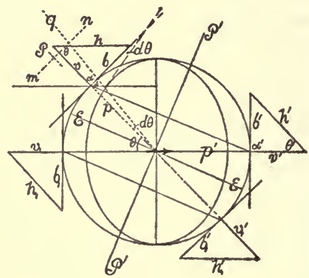


FIG. 17.

the float, was therefore virtually very light. The difficulty encountered resides in the fact that even small differences in the coefficient of expansion of BC and BD will seriously tilt the axis AC. To express these relations let h, v, b, be the hypothenuse, the vertical, and the base of a right-angled triangle as shown in fig. 15 and idealized in fig. 16. Let $\psi=dh/h=db/b$ for the same temperature increment of 1° C. be the coefficient of expansion of the base and of the brace (for convenience), and $\beta=dv/v$ that of the brass post. Then it follows easily that for an increase of 1° C. of the environment,

$$h^2\psi=v^2\beta+b^2\psi-bv\Big((\psi+\beta)\cos\alpha+\sin\alpha\, d\alpha\Big)$$

where α is the angle of inclination of the post and $d\alpha$ its increment. Hence, since $\alpha=90°$, very nearly,

$$d\alpha=(\psi-\beta)/\tan\theta$$

Since $\tan\theta=0.25$, nearly,

$$d\alpha=4(\psi-\beta) \text{ radians}$$

or, for the above $\alpha=0.3x$ seconds,

$$dx=2.7\times 10^6(\psi-\beta) \text{ cm.}$$

Hence, so small a difference of coefficient of expansion as $\psi-\beta=10^{-6}$ would give rise to a deflection of $dx=2.7$, nearly 3 cm. per degree of increase of temperature. In fact, this arrangement actually suggests itself as a remarkably sensitive method for measuring small elongations; for, since $d\alpha=(\psi-\beta)\cot\theta$, independent of all lengths, $d\alpha$ increases as θ decreases without limit, and the question is merely one of experimental adjustment. If the hypothenuse h alone expands, the remaining temperatures being kept constant,

$$d\alpha=2\psi/\sin 2\theta$$

or for the above data

$$\psi = 0\cdot35 \times 10^{-6} dx$$

quantities of the same order as the above. Thus if $dx = 1$ cm. (for the relatively small distance of scale from mirror, $D = 900$ cm.), $\psi = 0.35 \times 10^6$, and it should be easy to measure one-tenth of this expansion.

The peculiar interest which attaches to this equation for $d\alpha$ or any corresponding case is the absence of all need of length measurement in the combination. In the right-angled triangle *hvb*, fig. 15, it is merely the angle θ which must be given, all the quantities compared being numbers. Of course, the relation of x and α remains, into which the distance of the mirror from the scale will enter. In the complete equation (if $d\alpha$ is replaced by α)

$$\psi - \beta = \varphi x \tan \theta / 4D,$$

φ may be found directly as shown above.

If the interferometer is used, $x/4D$ is to be replaced by $\Delta N/2R$ so that

$$\psi - \beta = \varphi \Delta N \tan \theta / 2R$$

If values of the above order be inserted, *i.e.*, $\varphi = 0.01$; $\Delta N = 10^{-4}$; $R = 10^2$; tan $\theta = 0.25$; then $\psi - \beta = 12 \times 10^{-10}$. In case of the other equation, $d\alpha = 2\psi/\sin 2\theta$,

$$\psi = 12 \times 10^{-10}$$

In any case, therefore, an expansion of the order of 4×10^{-10} per vanishing interference ring ($\Delta N = 3/10^5$) should be measurable. This seems by far the most sensitive arrangement for measuring elongations which has yet been proposed. The full equation in question would be

$$d\alpha = \tan \theta \left(\frac{dh}{h} - \frac{db}{b}\right) + \cos \theta \left(\frac{dh}{h} - \frac{dv}{v}\right)$$

from which any of the above forms follow at once.

In its bearing on the horizontal pendulum, the above result is fatal. Braces of all types will have to be discarded. The following incidental experiments will bear this out: The brace was heated with a single rapid brush of the Bunsen flame, such as would not have imparted any easily appreciable increase of temperature to the massive rod. The times of observation were also recorded, the results being as follows:

Time.	x cm.	Remarks.
$9^h\ 28^m$	2.2	Zero position.
30	6.0	Quick brush of brace with Bunsen flame.
40	5.0	Cooling.
45	4.5	Do.
$9^h\ 45^m$	29.8	Additional brush of brace with flame.
55	23.7	Cooling.
10 15	14.0	Do.
26	11.4	Do.
42	9.5	Zero at 2.2 cm.

These results are quite regular. The extremely slow cooling shows how near the temperature is to that of the environment, the temperature excess being nearly negligible. It shows also how difficult it is to obtain rigorous temperature constancy in the metallic truss exposed to the surrounding atmosphere. Thus it would have taken considerably over two hours to dissipate the negligible difference of temperature in the last case.

Again, a Bunsen flame placed about 30 cm. from the brace, about at e in fig. 15, gave the following result:

$9^h\ 15^m$	$x = 1.5$ cm.	Cold. Burner placed as stated.
20	3.0	Gradually heating. Burner now removed.
28	2.2	Cooling.

The radiation of the Bunsen burner at a distance of about 1 foot is thus quite perceptible, in spite of the fact that the brace and standard are to some extent affected differentially.

With this experiment, therefore, the mysterious temperature variation has been cleared away, and it provides definite specifications for the installation of such an apparatus. I will only add that similar experiments tried on the scaffolding of the permanent mirrors produced only negligible effects. Thus a Bunsen flame run rapidly along any of the horizontal braces changes the deflection x only a few millimeters.

14. Inferences.—If we abstract from discrepancies introduced by the pendulum truss, which are to be separately treated, it may be assumed that the data obtained with the partially floating pendulum represent the actual tilting of the concreted subfloor of the laboratory. With a reasonably constant temperature in a cellar room, in the absence of artificial heat, temperature discrepancies should no longer be seriously menacing. To account for the difference between the small variations of α obtained in the absence of a float and the large variations on addition of the float to the same pendulum, it is sufficient to admit that the friction at the vertical pivots in the former case (float absent) was excessive, and that the full deflection can not appear, unless the weight is taken off the pivots as in the latter case (float present). This is particularly the case, since the inclination resulting from the expanding brace presently to be mentioned should in any case have been present. Capillary forces at the float, mounted axially as above, have produced no appreciable distortion, as they did when the water damper was mounted at the end of the beam. Finally, the float is itself a sufficient damper, and in the absence of air-currents the front of the case may actually be kept open, as was done in most of the later experiments, in a room free from artificial heat. Moreover, it does not seem necessary to construct the floating horizontal pendulum on so large a model as was done in the above paper, so that a smaller portable model may be a serviceable instrument for many laboratory purposes, seeing that the constants are determinable by the direct method indicated. In how far the sensitiveness may be increased by applying the buoyant force at the center of gravity or elsewhere can not be answered, as it is not unlikely that the capillary forces introduced will in such a case be a serious consideration.

The most interesting result obtained is the effect of temperature on the inclined brace supporting the vertical standard of the horizontal pendulum. It appears that even insignificant differences in the coefficient of expansion of these parts are at once manifested as an appreciable change of inclination varying with temperature. In fig. 14, for instance, if the large oscillations be interpreted as a temperature effect, one may estimate that the change of 1° C. of the temperature of the environment is equivalent to a deflection of $x=8$ cm. at the scale, or equivalent to about $\alpha=2.5$ seconds of change of inclination. It is therefore necessary to avoid this deficiency of the apparatus with scrupulous care; in other words, to avoid all lateral bracing if the material can not be guaranteed as rigorously homogeneous. Hence, in the more refined experiments, the pendulum is to be swung with advantage from a single sufficiently stiff metallic post anchored in the ground. Moreover, since the pendulum with a trough for the float supported quite free from the pendulum is virtually very light, a standard made of a length of 1-inch gas-pipe well anchored in the ground seems to be most promising. The case and the optical apparatus are in every instance to be supported entirely free from the horizontal pendulum.

The errors which have been detected in the case of the braced pendulum are in all probability also present in the case of a pier, if the pier confronts the illumination of the room or the heating-pipes of the building on one side only. In such a case, the exposed side will expand on the cold side as an axis, and a tilting of the pivotal line of the horizontal pendulum must result. Unfortunately, this is the condition to which the large pier in our laboratory is subject and which it is impossible to remedy. It is probable that the excursions observed with the steel pendulum in § 3 are largely to be interpreted in this way. Temperature observations will here be of little avail, since their distribution in the immense mass of masonry is in question. Similarly, the absorption and release of moisture when an unavoidably heated basement room passes from the damp summer to the dry winter conditions may have a similar tilting effect.

15. The precision measurement of elongations.—From another point of view the exceedingly sensitive expansion apparatus which has been described is interesting on its own account. In the diagram, fig. 16, the *hvb* triangle supports the horizontal pendulum *PP*, normal to its plane, on the pivotal hangers *pp; g* (outward from the plane of the figure) is the grating at the end of the pendulum, *n* and *m* the concave mirrors of the displacement interferometer. Apart from the instrumental (elasticity and viscosity, etc.) and environmental conditions, such an apparatus should register expansions dl/l of an order even smaller than 4×10^{-10} per vanishing interference ring, for the registered sensitiveness of the above apparatus could easily be increased. No doubt much of this would be taken up by the yield of the apparatus; but nevertheless it is over 10^4 times smaller than the expansion of an average metal per degree centigrade. The actual limit is necessarily an experimental question.

As an example it may be worth while to determine whether, potentially, the apparent contractions of rods lying longitudinally in the direction of the earth's motion could actually be measured and to what degree. With this end in view the problem may be stated with reference to fig. 17, where $P'P'$ is the polar, EE the equatorial diameter of the earth, p, p' two diameters in latitude 23.5°. The motion of the earth takes place along the diameter p' with a mean speed 3×10^{-6} cm./sec. The hvb triangle of fig. 16 is set up with the side v vertical in latitude 23.5° and the base horizontal and in the plane of the meridian as shown. The side v carries the horizontal pendulum P with its plane normal to the meridian, the line of observation being mn in the meridian. The excursions of the grating on the pendulum are read off on a linear displacement interferometer, the framework and the two component beams running in the same direction. All parts of the instrument are therefore identically subject to the same effects.

Twelve hours of rotation place the triangle in the opposed position $h'v'b'$, and the question to be determined is the change of angle α (nearly 90°) resulting, seeing that the relation of all the sides to the motion of the earth has been changed and v' instead of h moves parallel to it. For convenience in computation, the angle θ may be roughly taken as 45° instead of 47°, so that $h^2=v^2+b^2=2v^2$. In this case we may write

(1) $$d\alpha=2dh/h-dv/v-db/b$$

by reducing the equation for $d\alpha$ above. It should be noticed that the equation is purely numerical, the degree being zero.

Let v' be the velocity of the earth, c the velocity of light, so that $\beta=v'/c=10^{-4}$ and $\sqrt{1-\beta^2}$ is the longitudinal contraction coefficient. The size of the parts hvb and $h'v'b'$ under conditions of motion may therefore be replaced respectively by

$$\begin{array}{rlrl} \sqrt{2}v & \sqrt{1-\beta^2} & \qquad \sqrt{2}v & \sqrt{1-\beta^2/2} \\ v & \sqrt{1-\beta^2/2} & v & \sqrt{1-\beta^2} \\ v & \sqrt{1-\beta^2/2} & v & \end{array}$$

whence, nearly,

(2) $$d\alpha=-2\frac{\beta^2}{4}-\frac{\beta^2}{4}+\frac{\beta^2}{4}=\frac{\beta^2}{2}$$

On the displacement interferometer

(3) $$d\alpha=\varphi\theta=\varphi\Delta N/2R$$

where φ is the inclination of the axis of the horizontal pendulum in radians, ΔN the micrometer displacement at one of the opaque mirrors corresponding to the two positions, R the distance of the grating at the end of the horizontal pendulum from its axis. Incorporating equation (3) finally,

(4) $$\Delta N=\beta^2R/\varphi$$

which is the required equation.

In the apparatus used above, $\varphi = 10^{-2}$ and $R = 10^2$ cm., $\beta^2 = 10^{-8}$. These are moderate; the former could easily be reduced. Hence,

$$\Delta N = \frac{10^{-8} \times 10^2}{10^{-2}} = 10^{-4} \text{ cm.}$$

Even in the case of the present apparatus, therefore, about three interference rings should vanish (potentially) between the positions hvb and $h'v'b'$.

A similar comparison might be made for the position of the horizontal pendulum normal to the plane of the diagram, in relation to the first and final conditions discussed. But the lines are now oblique and require two or more projections, and this additional complication is superfluous here.

Contractions of the pendulum itself must be negligible, as these merely displace the center of gravity in the plane of the pendulum and are otherwise not amplified. Tidal forces have approximately the same value in the two positions, or may be allowed for. There remains, therefore, the contraction of the earth itself, which changes from a sphere of radius r to an oblate ellipsoid with its minor axis $r\sqrt{1-\beta^2}$ in the direction of motion p'. In a general way we may state at the outset that as the triangle is a part of the earth, its distortion could not be recognized for the lack of an independent base of comparison. But the question is advantageously approached, specifically, as follows. Fig. 17, which contains the sphere and the ellipsoid in question, shows that the diameter p is displaced to q and that the angle α moves over a distance

$$s = r d\theta / \sin\theta \text{ nearly}$$

where $d\theta$ is the angle between p and q and $\theta = 45°$ nearly. But the displacement s is the contraction of $r\cos\theta$ or

$$s = r\cos\theta \cdot \frac{\beta^2}{2}$$

Hence,

$$d\theta = \frac{\sin 2\theta}{4}\beta^2 = \beta^2/4, \text{ nearly.}$$

The same angular deviation occurs between the two tangents or bases prolonged, since the equation of the ellipse referred to the circumscribed circle is

$$x = r\sin\theta, \quad -y = r\sqrt{1-\beta^2}\cos\theta,$$

$$d\theta = \frac{\beta^2}{4}\sin 2\theta = \beta^2/4 \text{ nearly.}$$

Hence, the two angles, if

$$d\alpha = 2d\theta = \beta^2/2 \text{ as above}$$

seeing that in the position $h'v'b'$ the angle α does not change. The displacement from p to q, therefore, keeps the center of gravity in the normal plane in which these lines are traces and no effect could be recognized. Similarly if the horizontal pendulum were attached to a large massive vertical pendulum (rigid plumb-line) the displacement $d\alpha/2$ would escape detection. Nevertheless the potential possibility of the method, well illustrated by this example, seemed to make it worth while to endeavor to develop it, for there are other non-compensated micrometric deviations of the earth's diameter, to which it would be directly applicable.

16. Improved pendulum.—The suspension of the symmetrical pendulum was now modified so as to embody the suggestions contained in the above work. The two pivots were supported on a single post of 1-inch gas-pipe, sunk into a hole in the concreted subfloor of the basement and secured with plaster of paris. It was hoped, in this way, to obviate the possibility of temperature disturbances in their immediate effect on the pendulum. Naturally, the post was insulated from every other part of the apparatus, so that the pendulum was quite free and independent. Its tin case was adjustably supported on the iron scaffolding carrying the mirrors, while the tank for the float rested on an independent standard rising from the subfloor in question. It is improbable that short of a special brick pier for the instrument a more advantageous method of mounting could have been devised. It was therefore interesting to observe how the whole apparatus would behave, on transition from fall to winter conditions; *i.e.*, to find the effect produced on turning on the steam heat of the laboratory. The observations are given in the next paragraph.

17. Observations with the new pendulum.—These observations are given in fig. 18 in the usual way, the inclinations α, in seconds of arc, being constructed in their variations with time. To determine the constants of the apparatus the micrometer method, similar to the above (§ 11, end), was employed. The variation of x due to a twist of 12° of micrometer screw was found to be $x=27$, 26, 26 cm., or on the average 2.2 cm. per degree, whereas the former value was $x=3.1$ cm. Hence, since the other constants are the same as above (the distance apart of the pivots being 75 cm. and the pitch of the screw 0.125 cm., or 3.47×10^4 cm. per degree of arc)

$$\alpha=\frac{10^{-4}\times3.47}{2.2\times75}=2.1\times10^{-6}\text{ radian}$$

or $\alpha=0.43x$ second of arc, nearly. This constant was used in the reductions. In addition to the deviations x, the temperature of the room and the weather conditions outside were taken daily. The latter showed no consistent influence and will be disregarded here.

The first branch of the α curve (AA), from July 16 to August 23, shows an initial ascent until July 20 and thereafter a fairly uniform descent. The temperatures (as shown by the temperature curve, fig. 18) during the first half of the observation period might suggest some relation, but they quite fail to do this during the second half, where temperature in general rises and the α curve falls. The curves may be real and indicate a gradual settling of the ground during the whole period, modified by rains, etc.; or there may have been a gradual viscous yield of the support of the pendulum in its concrete base. Whatever be its nature, the pendulum fully recovers from this apparently continuous subsidence during the second period of observation (*BB*), between August 23 and September 28, omitting the gaps at *a* and *b*. The temperature observations (not drawn) show no relation to the α curve whatever. The curve being undulatory, it can not be referred to any persistent yielding or

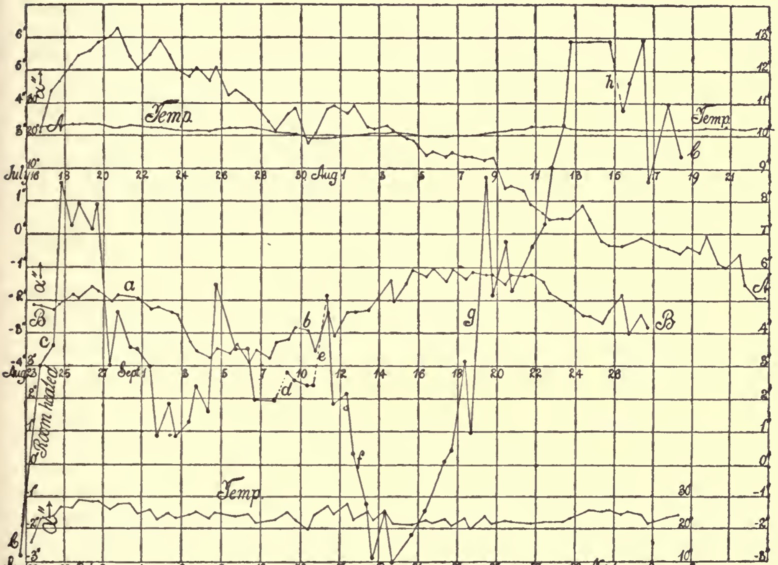


FIG. 18.

other similar discrepancy. It is probable, in fact, that both curves *AA* and *BB* show the actual tilting of the concreted subfloor of the laboratory.

On September 28 the steam heat was turned on (scale of α, on the right) and the totally new character of both the α curve *CC* and the temperature curve furnish abundant evidence of the importance of this disturbance. In fact, the pendulum behaves at first like an extremely sensitive thermometer. Observing that the temperature scale is enormously smaller, rise and fall of temperature, *i.e.*, depression and elevation of both curves, may in general be coördinated throughout. But there is no quantitative relation between the two curves. Thus the marked rise of temperature from about 16° to nearly 30° at the end of September and beginning of October shows an α effect in the curve following c quite inferior to the effect later between g and h, when the changes of temperature are much less marked. Moreover, the minimum between f and g is much too large to be associated with the mean temperature minimum, and in fact they do not coincide. The determined rise of the curve at g began much before any corresponding temperature change. The gaps at d and e introduce uncertainties, but nevertheless one would have expected an α minimum there. It seems probable, therefore, that temperature does not act directly on the pendulum (expansion of its parts) but acts on it through another system, which is probably the house itself. The marked effect of steam heat is to thoroughly dry out the basement room. It may be inferred that this is accompanied by redistributions of the stresses of the building and that the concrete subfloor responds to the alteration of load. It should be noted that the last curve CC has been dropped 7 seconds to accommodate it in the drawing and that therefore the α curve CC as a whole lies much above the original α curve AA, although the average temperatures are not very different. It seems improbable, therefore, since the curve has much more than recovered, in fact has considerably exceeded its original reading, at about the same temperature, that there can here be any viscous yielding in the apparatus itself. Further consideration will be given in the next section in connection with the steel pendulum.

Finally, since the inevitable variations here recorded are within 16 seconds of arc, it was out of the question to attempt to attach the interferometer apparatus, adapted for reading within hundredths of a second. The work was therefore abandoned.

PART II.—AN APPLICATION OF THE DISPLACEMENT INTERFEROMETER TO THE HORIZONTAL PENDULUM.

18. Introductory.—The displacement of ellipses or of interference lines in the spectrum is probably capable of being photographed for continuous registry, though less easily than the motion of a spot of light. At all events, it seemed interesting to endeavor to register the excursions of the horizontal pendulum by displacement interferometry, not so much with a view to recording seismological phenomena, as to approach by this means certain other problems, the tilting of the earth's surface relatively to the plumb-line, the measurement of the constant of gravitation, etc. The present paper, therefore, undertakes a new departure with this special end in view, with possibly some ulterior bearing on microseismology.

If the inclination of the axis of the horizontal pendulum is but a few degrees to the vertical and a large framework is in question (there is scarcely any limit to size other than strength of the material), the sensitiveness of the apparatus, when the excursions are read off in terms of light-waves, is astonishing; or at least it would be so if the instrument supplied with mirror and screen had not been so thoroughly perfected. The horizontal pendulum, moreover, has this peculiarity, that it is able to support relatively large weights; *i.e.*, relatively massive bodies may be subjected to each other's attraction.

19. Apparatus.—The horizontal pendulum has the usual form of a swinging gate and was constructed of ⅜-inch (vertical) and ¼-inch (oblique) thin steel tubes. The material available here was unfortunately slightly too thick-walled, a defect which will be modified in the future. Moreover, steel, as has been seen in the work with the electrometer, is an undesirable metal in the varying magnetic field of a city when the micrometry of angles is in question.

The frame of the pendulum, as shown in fig. 19, is very simple. *ABC* is the truss of steel tube, soldered at *A* and *B* and terminating in the brass clutch at *C*, into which it is also soldered. The tube *AB* is slotted at top and bottom and each end receives a solid cylinder, *a* and *b*, of glass-hard steel, snugly. These are held in place by collars *c* and *d*. The cylinder *b* contains a conical socket to receive the point of the horizontal steel pivot *t*, a portion of the tube *A* having been removed at this part. Similarly the cylinder *a* contains a vertical slot (or reëntrant dihedral edge) to receive the horizontal pivot *s*. These pivots are adjustable toward and from the rear, from right to left, and each is revolvable about a horizontal axis normal to the figure, in a way which will presently be shown. The distance between pivots was 97 cm., the distance between the cylinders *AB* and *D* about 111 cm., and the reduced end projects about 16 cm. beyond the edge *EE* of the brick pier to which the pivots are attached. *D*, clutched by *C*, is the hollow stem of the tablet *f*, which holds the plane dot slot arrangement to secure the grating *g*, a spring passing down the interior of the tube *D*. The lower pivot *t* should preferably point towards the center of gravity *G*.

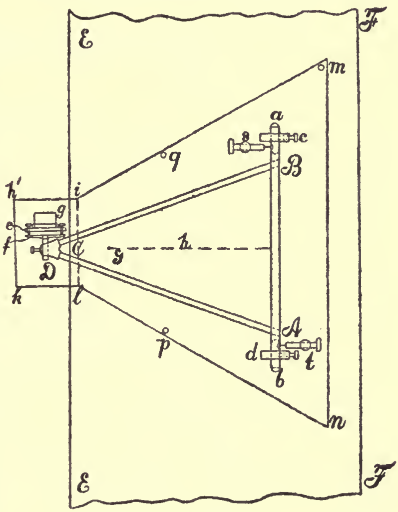


FIG. 19.

The whole apparatus is inclosed in a more or less triangular flat case *h'mnk*, firmly bolted to the wall at *q*, *m*, and *p*. The two sides of the case beyond the pier, *h'ilk*, may be slid off to the left, and then the whole remainder lifted off its bearings without touching the pendulum, as the case has no rear wall. The front face is within 3 inches of the face of the pier. This arrangement was found very satisfactory. The head of the case *ki* is of course glass-faced (identical plates) in front and rear, so that the grating, etc., may be seen. A convex mirror if placed at *g* reflects a beam of light, showing the pendulum to be nearly stationary during the day, in spite of the surrounding city. The slow normal variations were not greater than 5 mm. on a radius of 13 meters, corresponding therefore to about 40 seconds of arc. The corresponding change of inclination relative to the plumb-line would be less than one one-hundredth of this, depending on the period given to the horizontal pendulum.

The mass of the pendulum was 720 grams, that of the grating holder originally 475 grams, and that of the grating, etc., about 55 grams, making a total of 1,250 grams; but these masses are to be much modified in the future. The center of gravity, at *G*, with the grating in place, was originally about 80 cm. from the axis *AB*.

The grating at *g* moves between the two opaque mirrors, usually called *M* and *N*, of the displacement interferometer, in the way shown in my earlier work on interferometry.

But these mirrors *M* and *N* must in the present case be identically concave, silvered on their front faces, and at a distance equal to their common radius of curvature from the center of the ruled face of the grating. This center is illuminated by the impinging beam of light from the collimator, and the returned beams, reflected from *M* and *N*, must pass through the same area of illumination. In such a case the reflection at *M* and *N* is always normal to those surfaces and the rotation of the grating does not interfere with the definition of the ellipses of the interference pattern. For any other distance of *M* and *N*, except these radii of curvature, the spectra in the telescope will cease to coincide horizontally on rotating the grating and the ellipses would at once vanish. On the other hand, the displacement of the grating *in arc* at the end of the arm of the horizontal pendulum is registered in amount by the shifting of the ellipses in the interfering spectra. This displacement includes, of course (as a small correction), the additional thickness of glass introduced by the rotation of the grating. The displacement in question is the arc, which, when referred to the axis of the horizontal pendulum, measures its angular deviation resulting from the inclination of the earth's surface relatively to the plumb-line.

It is convenient to exhibit the details of the instrument (figs. 20 and 21) in separate parts for convenience in drawing, these being superimposed in practice.

Fig. 20 shows the attachment of the two opaque mirrors *M* and *N* of the interferometer to the pier *P*. Here *abcd* is an ordinary framework of ¼-inch gas-pipe. The end *a* is firmly plastered into the pier, *b* rises at a slight angle, *cd* being horizontal and parallel to the pencil of light from the slit, while *g* shows the position of the grating on the horizontal pendulum in fig. 19. The arm *b* lies below the case in that figure and is free from it. Each of the mirrors *M* and *N* is on plane dot slot adjustments, and *M* is provided with a Fraunhofer micrometer suggested in the figure. Both *M* and *N* can be rotated around horizontal and vertical axes for adjustment, the former *M* being provided with a fine motion. The clutch *e* and the corresponding one for *M* (not shown) allow the micrometer to be placed at a greater or less distance from the grating. The center of the mirror is about on the same horizontal level as the grating. It is also usually convenient to place the lens *L* of the collimator in its screen, on the same rod *cd*, with an appropriate clutch and rack and pinion.

The complementary framework is shown in fig. 21 and holds the slit *s* of the collimator (or the filament of a Nernst lamp) and the two telescopes *T* and *T′* in place for observation, *T* being used for the direct slit image and *T′* for the diffraction spectra and interferences. The framework *fghi* is, as before, of

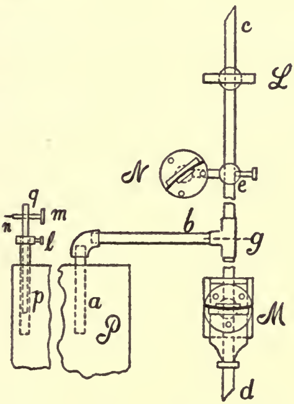


FIG. 20.

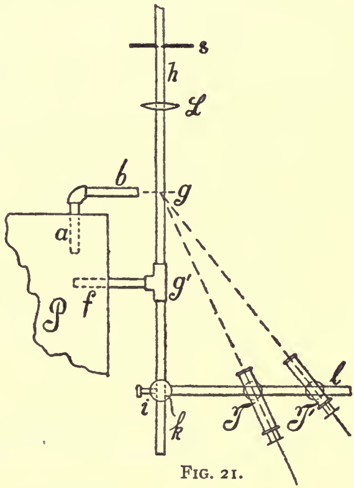


FIG. 21.

gas-pipe, *f* being firmly plastered into the wall on the front face of the pier (the other one, *ab*, being on the side). The telescopes *T* and *T′* are necessarily adjustable on a horizontal and vertical axis, and may be raised and lowered and moved right and left along the rod *kl*, held by a firm clutch at *k*. The lens *L* may also be carried on *ab*, as has been stated. Right-and-left, up-and-down motion is needed for the insertion of these appurtenances. The rods *cd* and *hi* are not in the same horizontal or the same vertical plane, so that the systems may be superposed as stated.

In the course of the work it appeared, however, that the framework of simple pipe was annoyingly subject to tremors. It was found necessary to lengthen the rods *gd* and *gc*, *g′i* and *g′h* to over a meter in length. Hence it was preferable to bolt the pairs of parallel rails together for increased stiffness and to secure the ends of each pair with a wide *tetrahedral* brace of gas-pipe, abutting at the pier, against horizontal and vertical displacement. So adjusted, the system was light and rigid and easily modified for the different purposes of the experiment. The additional braces have not been shown in the figure, as they depend on purely local conditions, the base of each tetrahedron being at the pier and its apex at the corresponding common ends of the pairs of rails, *gc*, *g'h*, *g'i*, and *gd*. All appurtenances like lenses, mirrors and micrometers are attached with strong removable clamps, provided where needed with rack-and-pinion attachment for focussing, etc. This allows of an easy and indefinite modification of the sytem and is thus very convenient for experimental purposes of the present kind. In the later work the telescope rod *kl*, fig. 21, was discarded in favor of a tripod standing on the floor.

It is finally necessary to describe the pivots of the horizontal pendulum, and these are also given in fig. 20. Here *p* is a length of ¼-inch gas-pipe fixed in the wall with plaster. The outer end is split lengthwise and carries a collar and set-screw *l*, so that the brass rod *q* fitting the pipe *p* snugly may be firmly secured. The end of *q* carries the horizontal, very snugly fitting screw *m* of ¼-inch brass, which is tipped at *n* with the steel point of a darning needle. The point of *n* is received by the socket of the horizontal pendulum. Thus *n* may be rotated about *qp* and moved fore and aft or right and left for adjustment. The socket is a conical hollow of about 60° and of glass-hard steel.

20. Equations.—With regard to the apparatus just described, the size of which was limited to conveniently fit the given pier, the following equations may be used to obtain an estimate of the sensitiveness to be expected.

Let φ be the inclination of the axis of the pendulum to the vertical and θ an angular excursion of the pendulum, measured from its position of equilibrium. Let h be the normal distance of the center of gravity from the axis. The rise of the latter above its lowest position is

$$(1)\qquad y=h(1-\cos\theta)\sin\varphi=2h\sin\varphi\sin^2\frac{\theta}{2}$$

and the energy potentialized, if the total mass is M, will be

$$(2)\qquad W=2Mgh\sin\varphi\sin^2\frac{\theta}{2}$$

which for small displacements corresponds to the torque Fh at the angle θ. This torque is

$$(3)\qquad \frac{\partial W}{\partial\theta}=Mgh\sin\varphi\sin\theta=Mgh\,\varphi\theta \text{ nearly}$$

or the total force F acting at center of gravity, or F/M per gram of mass M is

$$(4)\qquad \frac{F}{M}=g\varphi\theta$$

In the above apparatus $M=1{,}245$ grams, $h=80$ cm. Hence per vanishing interference ring, *since the grating moves*, if ΔN is the displacement of the micrometer to bring back the center of ellipses to the fiducial sodium line

$$(5)\qquad \theta=\frac{\Delta N/2}{R}=\frac{\Delta N}{2R}$$

where R is the distance of the point of the grating at the line of light corresponding to the slit, from the axis of rotation. In the apparatus $R=111$ cm.

Hence the angle corresponding to a vanishing ring is, since $\Delta N = 30 \times 10^{-6}$,

$$\theta = \frac{30 \times 10^{-6}}{2 \times 111} = 13 \times 10^{-8} \text{ radian} = 0.028''$$

Furthermore, if $\varphi = 1° = 0.0175$ radian

$$F/M = 981 \times .0175 \times 13 \times 10^{-8} = 2.3 \times 10^{-6} \text{ dyne}$$

per vanishing ring per gram-mass at the center of gravity of the pendulum.

The total pull of the center of gravity of the above pendulum is thus

$$F = 2.3 \times 10^{-6} \times 1245 = 2.9 \times 10^{-3} \text{ dyne}$$

per vanishing ring, on one side. By lengthening the radius from h to R this may be decreased to about 2×10^{-3} or less. Hence in case of gravitational attraction at one centimeter of distance it would require two equal masses m (since $\gamma = 6.7 \times 10^{-8}$ roughly) of the value $m = \sqrt{20 \times 10^4 / 6.7} = 1.8 \times 10^2$ grams, or 180 grams per vanishing interference ring, at a distance of 1 cm. On the other hand, the framework of the above pendulum is unnecessarily heavy, and was constructed out of the material at hand. It could easily be reduced in weight much below the above datum, or the greater part supported on a float, so that the case may be stated many times more favorably.

Resuming equation (3), if K is the moment of inertia, i the radius of gyration, and T the period and l the length of the *horizontal* pendulum,

$$T = 2\pi i \sqrt{\frac{I}{gh\varphi}} \tag{6}$$

an equation from which φ may be found in terms of T, i, and h which must be measured.

Again, the indicated length H of pendulum (distance from the center of gravity to the point of intersection of the axis and the plumb-line through the center of gravity) is

$$H = h/\sin\varphi = h/\varphi, \text{ nearly.} \tag{7}$$

The change of vertical inclination α of the axis of the pendulum corresponding to the horizontal deviation θ is, then,

$$\alpha = \frac{h\theta}{H} = \theta\varphi, \text{ nearly,} \tag{8}$$

or if the period T be introduced from (6) and θ from (5)

$$\alpha = \frac{4\pi^2 i^2}{T^2 hg} \frac{\Delta N}{2R} \tag{9}$$

It is in equation (8) that the condition of remarkable sensitiveness resides. Thus, if the interferometer is used, $\alpha = \varphi\Delta N/2R$, and, if $\Delta N = 30 \times 10^{-6}$ and $\varphi = 10^{-2}$ (somewhat less than 1° of arc), $R = 111$ cm., as above,

$$\alpha = 13 \times 10^{-10} \text{ radians} = \cdot 00028''$$

per vanishing interference ring.

If F, as before, is the force at the center of gravity, the corresponding force at the grating, a distance R from the center, is

$$F_R = Mg\varphi\theta\frac{h}{R} = M\frac{4\pi^2 i^2}{T^2 R}\theta \tag{10}$$

since φ is given by equation (6). This equation implicitly contains h, since i refers to an eccentric axis and $i^2 = i^2_0 + h^2$; but i may be found directly.

The deviation θ is given by (5). If, however, the device* of two parallel mirrors, equidistant (distance R) from the axis of the horizontal pendulum, be used, and if light impinges on either mirror at an angle of incidence I (the impinging and reflected beams being always parallel),

$$\theta = \frac{\Delta N'}{2R \sin I} \tag{11}$$

where $\Delta N'$ is the displacement of the micrometer. The horizontal pendulum is in this case constructed symmetrically to the vertical axis in the form of a balance beam, but somewhat heavier on one side.

Finally, the compound pendulum may be supported on a cylindrical float, symmetrical or not to the vertical axis of the pendulum and submerged in water or some other liquid. In such a case, the mass of the compound pendulum may be reduced in any degree without serious difficulty from capillary forces, as will be shown below. If the center of buoyancy is in the vertical line passing through the center of gravity of the horizontal pendulum, the above equation needs but slight alteration. Let V be the volume of the float, so that $V\rho g$ is the buoyancy. Apart from the temperature conditions, $\rho = 1$, and hence the equations take the successive forms, since $(M-V)g$ is supported at the center of gravity, instead of Mg,

$$W' = \frac{1}{2}(M-V)gh\varphi\theta^2 \tag{12}$$

$$T' = (M-V)gh\varphi\theta \tag{13}$$

The force at a distance R from the axis is, when the center of gravity is at a distance h,

$$F'_R = \big((M-V)g\varphi h/R\big)\theta \tag{14}$$

Hence the force has been reduced in the ratio of $M/(M-V)$ for the same θ. One may also note that it is smaller, not only as φ is smaller, but as h/R is smaller. Hence a symmetrical form of pendulum, like the balance-beam, but slightly heavier on one side, suggests itself for work on gravitational attraction, etc. It was not found difficult to reduce the weight of the pendulum by flotation to 40 grams, *i.e.*, about 31 times. Hence the force per vanishing interference ring computed above would now be

$$F'_R = 2\times 10^{-3}/31 = 6\times 10^{-5} \text{ dynes, roughly.}$$

This would be equivalent to the attraction of two 30-gram weights at 1 cm. of distance.

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* Barus: Am. Journ. Sci., XXXVII, pp. 83 et seq., 1914.

Furthermore,

$$T' = 2\pi i \sqrt{\frac{1}{gh\varphi}\;\frac{1}{1 - V/M}} \tag{15}$$

whence, since $\theta = \Delta N/2R$

$$F'_R = \frac{M}{R}\,\frac{4\pi^2 i^2}{T^2}\theta = \frac{M}{R^2}\,\frac{2\pi^2 i^2}{T'^2}\Delta N \tag{16}$$

all of which quantities are easily determined with accuracy. To find the radius of gyration i, for instance, a body of known moment of inertia may be suspended at the end of the horizontal pendulum and the periods T of the pendulum before and after suspension determined, with or without the float. Finally the change of vertical inclination α becomes

$$\alpha = \frac{h\theta}{H} = \theta\varphi \text{ (nearly) } = \frac{4\pi^2 i^2}{T'^2}\,\frac{M/(M-V)}{Rgh}\Delta N \tag{17}$$

If the pendulum is damped, which will usually be the case, it may be necessary to observe the logarithmic decrement, in order to compute the free period in the usual way.

If the buoyant force due to the float does not pass through the center of gravity of the solid parts of the pendulum, but at a distance h' from the vertical or pivotal axis, the new distance of the center of gravity h'' when the pendulum is partially floating will be

$$h'' = \frac{Mh - Vh'}{M - V}$$

Hence, if $h' = h$, then $h'' = h$, resulting in the equations just deduced. But if $h' = 0$, *i.e.*, if the buoyant force passes through the point of the lower pivot,

$$h'' = \frac{M}{M - V}h$$

Thus the equations deduced become identical with the original equations (2) *et seq.* The float therefore adds nothing to the sensitiveness except in so far as it removes friction at the pivots and supplies a reliable damper for the pendulum. It is in this form that the float will be applied below. Since the torque equation is now again

$$T = Mgh\varphi\theta$$

where all references are to solid parts of the pendulum, h may be accurately found by placing weight m at a distance l from the plane of the pendulum, or better, by placing weights alternately before and behind this plane, at a distance l apart. The torque applied is then $T = mgl$, whence

$$h = \frac{ml}{M\theta} \tag{19}$$

This method will be used effectively in several experiments below. It is an excellent test on the reliability of the damper, since h can also be determined directly by the suspension of the solid beam of the pendulum. In the adjustment adopted, at a scale distance of 900 cm., $ml = \text{gram} \times \text{cm.}$ on the scale-pan, produced a deflection of about 1 mm.

A few other equations of minor importance may be added. If the indicated length is H and the horizontal pendulum be treated as a vertical pendulum of length L, the point of suspension being the intersection of the plumb-line through the center of gravity and the line determined by the points of the two pivots, the observed period is

$$T=2\pi\sqrt{L/g} \text{ and } I^2=LH \tag{20}$$

where I is the corresponding radius of gyration.

If the end of the horizontal pendulum is loaded with the weight m of a disk at a mean distance R from the axis for the measurement of gravitational attraction, since $(M+m)\ h'=Mh+mR$, the new force at R is

$$F'_R=F_R\left(1+\frac{m}{M}\ \frac{R}{h}\right) \tag{21}$$

When the end of the pendulum is similarly loaded for the determination of its radius of gyration, since

$$i'^2=i^2+mR^2/M \tag{22}$$

the new period is

$$T'=T\sqrt{\frac{1+\frac{m}{M}\ \frac{R^2}{i^2}}{1+\frac{m}{M}\ \frac{R}{h}}} \tag{23}$$

Since T' and T are observed and m, M, R, h given, i may be computed. The horizontal pendulum itself thus supplies the value of i.

If the lower pivot is provided with a strong micrometer screw, by which it may be moved over a small distance z to the front or rear of the plane of the pendulum, the computed value of α may be tested independently. Thus let y be the distance apart of the pivots and z the displacement of the lower, then

$$\alpha=z/y=\varphi\theta=\varphi x/4D \tag{24}$$

when in the method of deflection x is the increase of the distance apart of the two images of the slit, at a distance D from the further mirror. Hence

$$\varphi=4Dz/xy$$

where φ must agree with its corresponding datum from the pendulum measurement in terms of period. Thus, since φ and h may be obtained independently, the torque T, etc., is given independently. This method will also be applied below.

21. Observations with a grating rotating on a fixed vertical axis.—When the opaque mirrors M and N are identically concave and are put on the ordinary interferometer at a distance equal to their radius of curvature from the stationary grating, the latter may be rotated (without translation) as far as the breadth of the opaque mirror N permits, without readjustment. The ellipses are not lost. Inasmuch, however, as different thicknesses of glass are introduced into the rays when the grating is rotated, the ellipses travel horizontally through the spectrum from the red to the violet end or the reverse.

They are about equally clear in all positions. A displacement at the mirror N of about 4 cm. per meter, *i.e.*, 0.04 radian, equivalent to a rotation of 2.3° of the reflected ray, or a rotation of 1.15° for the grating, was within the scope of the interferometer and the tests were made within this limit. It is far in excess of anything required in the horizontal pendulum. No doubt if the mirror N had been wider, the ellipses could have been retained for larger angles of rotation of the grating, though they would in such a case travel several times through the spectrum. The micrometer at M would have to be used.

If long columns of glass are to be inserted in either beam (GM or GN) the concave mirror is not available, since the direct slit images will then have different focal positions. The rays issue from the plane-parallel column, parallel to this focal direction, but from a virtual focus nearer the concave mirrors. Hence, if the column is placed in the beam GM, the beam GN will, as a rule, have to be correspondingly shortened. The algebraic relations are complicated.

22. Observations with the interferometer.—The horizontal pendulum with which the following observations were made had the following constants, M being the total mass of the fixed parts, m the attached mass, h the distance of the center of gravity from the axis, R the distance of the vertical line of light on the grating (also mean distance of m and of F_R) from the axis, φ the inclination of the axis: $M=1,250$ grams; $m=227$ grams; $h=80$ cm.; $R=111.3$ cm. The observed periods (primes refer to the loaded pendulum) for M and $M+m$ were $T=18.48$ seconds; $T'=18.87$ seconds. Thus $i=85.1$ cm.;

$$\varphi=\alpha/\theta=0.01081 \text{ radian}=0.62°$$

and $H=7,394$ cm.; $L=8,488$ cm.; $H'=7,834$ cm.; $L'=8,853$ cm.

Since $\theta=\Delta N/2R$ when ΔN is the mean displacement for the horizontal deflection (θ) of the pendulum,

$$\alpha=10^{-5}\times 4.86\Delta N \text{ radians.}$$

Thus, if $\Delta N=10^{-4}$ cm., $\alpha=10^{-3}$ second of arc, or the change of α per vanishing interference ring ($\Delta N=10^{-6}\times 30$) is 0.000310 second of arc. Since T may easily be increased over 3 times, this limit may be reduced to $\alpha=.000030''$ per ring.

Similarly, the forces at distance R from the axis of the horizontal pendulum are

$$F'=F\left(1+\frac{m}{M}\,\frac{R}{h}\right)$$

$$F_R=42.9\,\Delta N;\quad F'_R=53.7\,\Delta N$$

Thus if $\Delta N=10^{-4}$ cm., $F'_R=0.0054$ dyne or about 0.0016 dyne per vanishing interference ring, in case of the pendulum loaded with the disk m.

In the graph which follows an example is given of a series of observations made for θ and α, and no further explanation will be needed. Since $\alpha=\varphi\theta=0.01080$, α need not be recorded.

In fig. 22 these observations have been inscribed, the ordinates being the inclination of the pier α, in hundredths of a second of arc, very nearly. It will be seen that the inclination increases as a whole from the beginning to the end of the month, the total range lying within something over 2 seconds of arc. The rise is particularly marked and sustained after the 14th, and the difference of inclination between the first and second halves of the month is about 1 second.

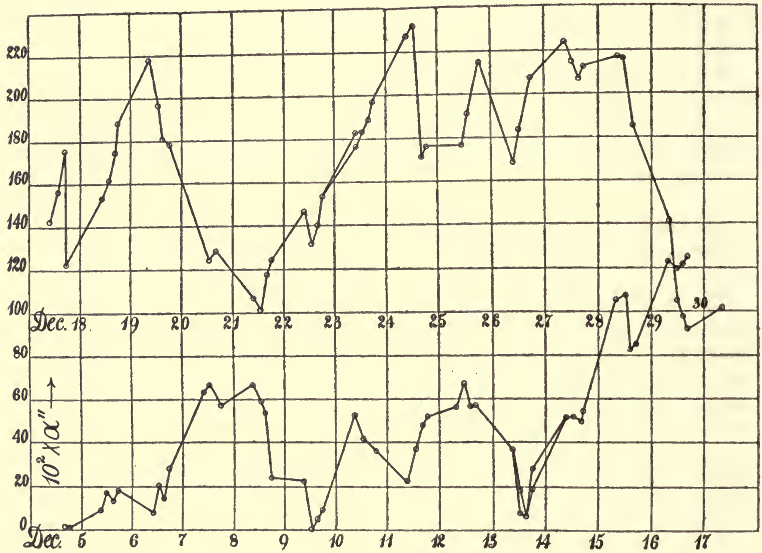


FIG. 22.

As the observations were made in an unavoidably steam-heated room, it is probable that the flexure of the pier, etc., due to thermal causes, has been largely operative in modifying the trend of the curve; for on comparing the curve as a whole with the thermostat sheets (not shown) a retarded effect is possibly suggested, such as one would suspect if variations of surface temperature should penetrate massive masonry. It would then be possible for the curve to have different heights at the same temperature. Naturally such comparisons are very vague, and it is the range of values of α admissible in the apparatus which is here of paramount interest. Furthermore, as the hill on which the laboratory stands is, at present, being tunneled, so that the building is subject once or twice a day to the tremors resulting from the vigorous blasting underground, adequate conditions for the installation of an apparatus of the present kind are still remote. It is really surprising that interferometer observations could be made, without essential difficulty, under these circumstances. During an explosion, of course, the ellipses vanish, to reappear, however, immediately afterward, sometimes with displacement, such, for instance, as is indicated by the doubled parts of the curve. The use of the water damper, moreover, which was necessary here, is objectionable, though it has not, probably, introduced any marked error into the observed curve (see doubled parts). Finally, the use of a steel horizontal pendulum with its plane in the magnetic meridian is inadmissible. I have not, therefore, endeavored to interpret the results, but they are given simply as an example of a systematic series of observations, extending over a month. I hope in the summer to resume the work in the absence of the annoyances referred to.

I may add in conclusion that the experiments referred to above, for measuring the gravitational attraction of two identical brass disks, led to curious results. It is easily seen that for constant mass the attraction of nearly contiguous disks should increase roughly as the fourth power of their radius. For disks 20 cm. in diameter, however, the result is an invariable *repulsion*, several times as large as the estimated gravitational attraction, the position of equilibrium being reached gradually in the lapse of several minutes. The subject will be systematically discussed in Chapter II.

23. Further observations. Film grating. Oil damper.—After the above experiment, the steel horizontal pendulum was used for other purposes and observations on the tilting of the pier were discontinued. Later, however, the apparatus was again available and a variety of experiments was made with it. In the first place, the water damper was replaced by an oil damper, as it seemed probable that the surface tension of illuminating oil and its slower evaporation would be an advantage. Under like conditions (though it proved sufficiently serviceable) it did not check the vibration as effectively as the water damper. The modification of chief interest, however, was the insertion of one of Mr. Ives's film gratings (in the usual double plate-glass protection) in place of the plate-glass grating. The film grating in question had about 15,000 lines to the inch, so that the dispersion was excessive, the ellipses being large and diffuse and with a long horizontal axis. To obviate this difficulty a thick compensator was introduced into the component beam M passing to and from the micrometer. For this purpose three thick plates of glass were cemented together with Canada balsam to a combined thickness of something over 2 cm. The ellipses now became adequately sharp and almost circular in form. In consequence of the multiple reflections described in Chapter IV, Part II, the ellipses are not so strong as in case of the grating ruled on plate glass, and they are much harder to find; but they are nevertheless quite serviceable. The single-plate film grating of § 60 was not at hand at the time. It is advisable to try out the double-plate film grating first on the fixed interferometer, in order to determine which lines of the individual images of the slits are to be placed in horizontal and vertical contact, together with the distance which corresponds to the different interferences on the micrometer. After this is done, the corresponding adjustment of the interferometer is easier.

It is also advisable to adjust the plate-glass grating in both cases for comparison. In figs. 23, 24, 25, 26 an example is given of these observations in the usual way. Data between January 15 and 24 (fig. 23, *A*) and March 3 and 6 (preceding fig. 23, *B*) exhibit the behavior of an oil damper with the ruled grating. In the observations after March 14, running as far as July 14, 1914, the ruled grating was replaced by the film grating. Inasmuch as no essential change was made at the steel horizontal pendulum, the constants may be taken to be the same as above, viz, $M = 1{,}250$ grams; $h = 80$ cm.; $R = 111$ cm.; $T = 18.48$ sec.; $i = 85$ cm.; $\varphi = 0.0108$ rad. $= 0.62°$. Thus $\alpha = 10^{-6} \times 48.6\ \Delta N$ rad. $= 10 \Delta N''$, nearly.

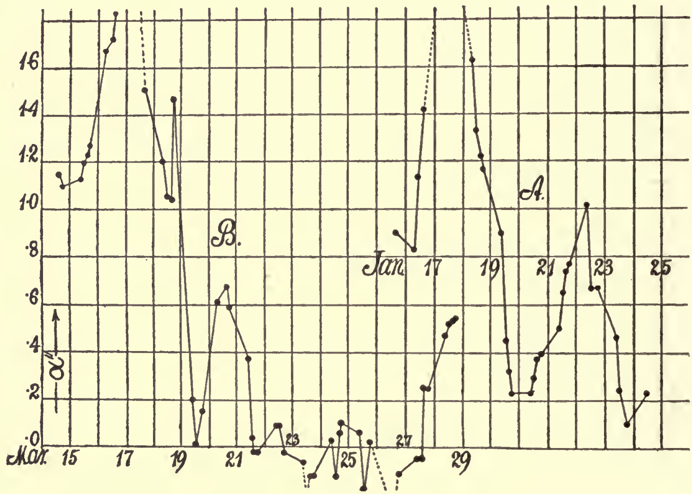


FIG. 23.

24. Inferences.—The curves in fig. 23, *A*, *B*, are independent so far as zero of measurement is concerned, but they already exhibit a tendency to decline in the direction of a decrease of α. This was pronounced in January and also in March. It is not a regular decrease, so that the cause can hardly be, or at least not wholly be, sought in the yield of the parts of the apparatus; for in such a case there would be no recovery (increase of α), a feature which is often marked. The continuous observations (*i.e.*, with the same uninterrupted zero) are given in figs. 24, 25, 26. The same scale is used throughout, but on April 22 and May 15 it was necessary to displace the graph in order to accommodate the observations on the sheet. The amount of displacement is shown. Here also there is a gradual and continuous decrease of the values of α. Beginning on April 1 with α about $2''$, the observations pass through a succession of oscillations to the lowest value of α recorded, about $-4.2''$, on May 18. After this there is intermittent partial recovery, so that on June 28 α has risen to $-1''$. The decline, however, at once commences, and on July 8 α is about $-3.3''$, from which it rises to $-2.4''$ at the end of the work. The maximum range of the tilting of the pier, α, between April and July is thus about from $+2''$ to $-4''$, or $6''$ of arc.

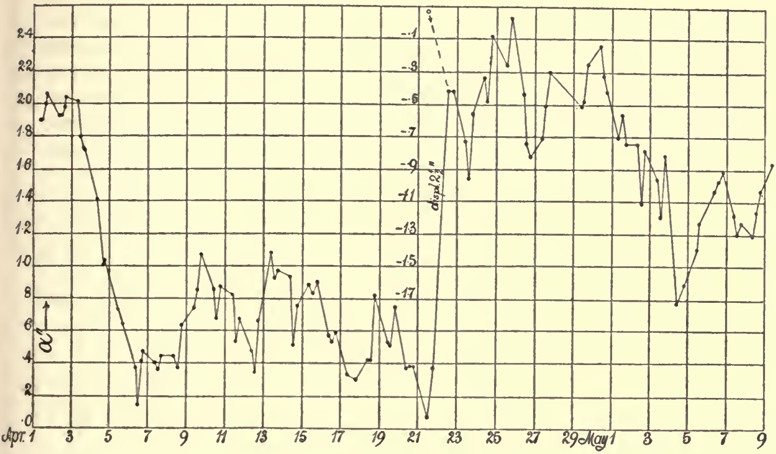


FIG. 24.

FIG. 25.

Between the oscillations proper, which are as a rule sharply marked, both at the maxima and the minima, there are regions of relative constancy of inclination. Thus between April 9 and 21 the inclination is nearly constant and about 0.6″; between May 17 and 28 there is a slow ascent from about −4.1″ to −3.6″; between June 1 and 12 a slow descent from −3.0″ to −3.4″; between June 16 and 25, relative constancy at about −2.8″, etc. Regarding the observations as a whole, it is not impossible that there may have been some slow yielding (quiescent frictional forces probably yield viscously) of the mechanical and the optical parts of the apparatus. The main features of the diagram, however, are due to the pier itself, or the pendulum, in responding to actual forces. On these the former errors may have been

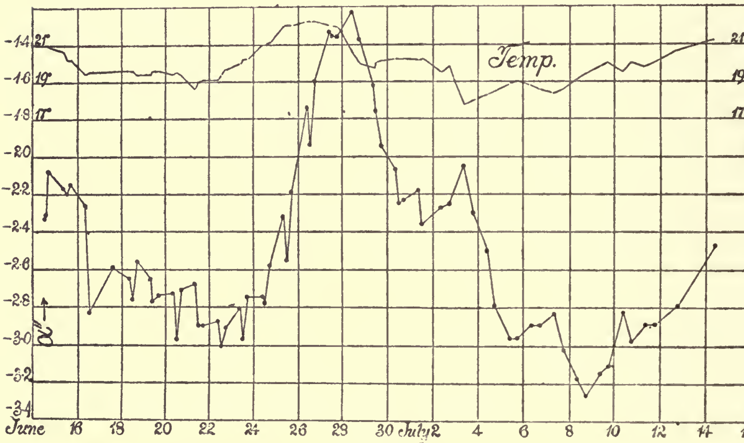


FIG. 26.

superimposed. Certain large drops on May 15 remain unexplained. The decline after April 3 is large and slow, so that it could be observed during its occurrence. This may therefore have been actual and not due to displacements of the pendulum resulting from subterranean shocks.

The observations were continued into June and July, with the expectation that when the basement room was no longer heated, the variation of α would practically vanish; but this is not at all the case, as the play of α in June and July scarcely differs from the average run of values during the winter months.

From the long-range point of view, inclination, α, decreases from about April 1 to about May 19, after which it increases intermittently again, recovering on June 29 and again on July 27 (not shown), about one-half the total decrement. It would not be difficult to arrange the minima in semi-monthly periods, if any reason for such large variations of α could be assigned. They are of course enormously above anything to be anticipated from tidal influences. So the maxima could be placed at March 17, April —, May 9, June 15, July—, March 29, April 27, May 29, June 28, July 27, in which the monthly periods are at least marked. But here again any adequate cause for such behavior has not been found, nor in any case has it been possible to separate the true from the adventitious tilting record.

As the pendulum was a thin steel tube and the direction north-south, one might infer changes of the earth's horizontal intensity. It is hardly probable, however, since whatever magnetization was present was induced by the earth, that forces of the required intensity could be present. The mechanical force at the grating for the displacement ΔN would be, roughly, $F=43\Delta N$. Since 1″ of arc of α was about $10\Delta N$, the mechanical force in question is thus 4.3 dynes per second of α. Such forces are not liable to be of magnetic origin.

Finally, if we compare the run of air temperatures given after May 28, for instance (the thermostat sheets were not accurate enough), though there is no detailed resemblance in the two graphs, some relation is none the less apparent. Thus the fall of temperature up to June 10 and its rise through a maximum on June 14, to fall again to June 22, is followed by the pendulum graph with a lag. So also the next temperature maximum on June 27 is followed by a pendulum maximum. This lagging of the inclination of the massive pier is precisely what one should expect if the observed oscillations are of thermal origin. It would seem that the parts of the pier exposed to the light expand and contract on the more equally temperatured colder parts, as an axis, as it were. The result would be a pendulum mechanism, very similar to the triangular bracket which I have discussed above, § 13, and which is peculiarly sensitive to the elongation of its parts. The expansion of any side of a triangle produces relatively marked tilting of the axis when the instrument of detection is a horizontal pendulum.

Taking the observations as a whole, there seems thus to be very little opportunity in the case of an ordinary massive pier of conducting observations, when fixity of inclination within 1″ of arc is in question, even for brief periods of time. Thus even after June 28, in case of the observed pier, there are changes of α amounting to 2″ of arc in ten days, and 0.2″ of arc per day must be looked upon as no unusual occurrence.

25. Improved aluminum pendulum. Observations.—The outstanding question bearing on the above observations was the possibility of a magnetic influence in case of the horizontal pendulum made of steel tubing, the pendulum being otherwise admirable because of its relative strength. A new pendulum, built entirely of aluminum tubing, with the exception of the brass clutch and the vertical hard-steel bearings for the pivots, was therefore installed. The aluminum tubes were screwed firmly together, the large triangle having the following dimensions and constants: Mass of pendulum, 554 grams; mass of grating holder and leveler, 456 grams; mass of (single-plate) film grating, 114 grams; mass of damper, 60 grams. This brings the total weight up to 1,124 grams, not differing much from the above. If lightness is an object (small torques being in question), the clutch and grating holder should also be made of aluminum and a lighter grating attached; but this is a secondary consideration here, though the mass might easily be brought down to about 700 grams.

The distance of the center of gravity from the axis was $h=93.1$ cm.; distance of line of light at grating from the axis, $R=110$ cm. The period was about as above, $T=19$ sec. The distance apart of the pivots, $p=97.1$ cm.

Though, in general, the aluminum triangle was a copy of the steel triangle, some improvements in construction were introduced. Thus a micrometer attachment was added to the lower pivot, so that a direct value of φ, the inclination of the pendulum axis, could be obtained. Windows were put in the case and both pivots were now accessible without removing it. The micrometer did not work as well as was expected, for reasons which did not appear. In several series of experiments, the mean of the horizontal angle corresponding to 30° of rotation of the micrometer screw of 32 threads to the inch and a distance of 97.1 cm. between pivots was

$$\theta=\frac{6.5}{820}=0.00793 \text{ radian}$$

since the reflected spot of light traveled 6.5 cm., when the scale distance was 410 cm., for each step of 30° of the micrometer screw. The corresponding change of inclination of the pendulum axis would correspond to one-twelfth of the pitch of the screw and would be

$$\alpha=\frac{2.54}{32}\,\frac{1}{12}\,\frac{1}{97.1}=68.1\times10^{-6} \text{ radian}$$

Since $\alpha=\varphi\theta$,

$$\varphi=\frac{68.1\times10^{-6}}{7.93\times10^{-3}}=8.6\times10^{-3}$$

not differing much from the corresponding value in case of the steel pendulum and there found by oscillation measurements, the pivots having been replaced as nearly as possible in their former positions. The projected horizontal distance apart of the pivots is thus about

$$97.1\times8.6\times10^{-3}=0.83 \text{ cm.}$$

which could easily be decreased and the pendulum made more sensitive (possibly ten times). Moreover, by using a fine wire plumb-bob, the angle φ could even be roughly measured by a Fraunhofer micrometer, showing the distance between plumb-line suspended from the point of the upper pivot and point of the lower pivot.

A "single-plate" film grating (see § 60) was mounted at the apex of the pendulum triangle. The interference rings were quite strong and clear and found without difficulty. At the outset it is possible that some yield of metallic parts may be registered, though the yield of the aluminum tubing, being in the plane of the pendulum, should not affect its reading appreciably.

The results with the aluminum pendulum are constructed in fig. 27, the curves showing the variation of inclination in seconds of arc in the lapse of time, here $\alpha = 8.6 \times 10^{-3}\theta$ radian. Since $\theta = \Delta N/2\,R = \Delta N/222$,

$$\alpha = 3.9 \times 10^{-5} \text{ radian} = 8\Delta N \text{ seconds of arc,}$$

the present factor 8 replacing the above value 10. As in the case of § 17, Part I, above, the present results are intended to test the variations from fall to winter conditions, and during the introduction of steam heat into the laboratory. The temperature observations are therefore also inserted in the diagram, so far as necessary. No difficulty whatever was experienced with the film grating throughout the whole of the work.

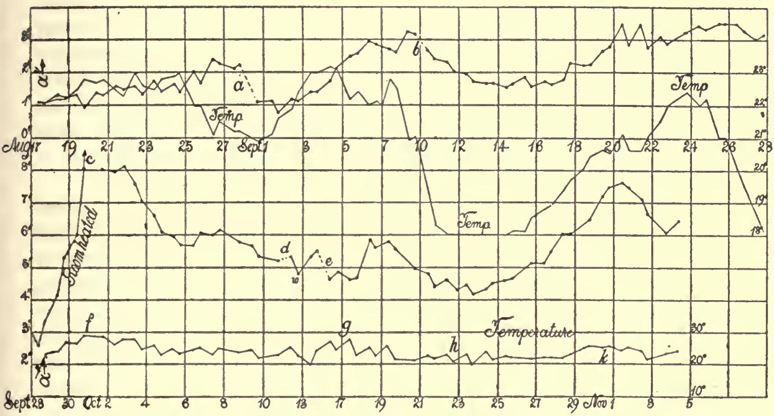


FIG. 27.

In the first half of the observations (upper curve), between August 17 and September 28, the curve shows a persistent upward trend. Gaps occur in the curve at α and b, owing to the absence of the observer, and these places happen to be associated with variations in the curve; but this is purely incidental. The curve fails to get back to its original reading. It seems probable, therefore, that the cause of this uniformly progressive march is the viscosity of the aluminum tubing out of which the pendulum was built, and what is observed is in the main a continuous viscous yield of the pendulum to the load of its own weight.

The lower curve shows the results between September 28, when the steam heat was turned on, and November 4. The effect of the sudden appearance of steam heat is sufficiently startling, as the curve on October 1 runs off of the scale at c. It was then necessary to displace the micrometer; but this was done so as to change the fiducial zero as little as possible. Afterwards, however, contrary to expectations, the curve again approaches its old value, so that the displacement would not really have been necessary. What took place was probably something like this: the access of heat in the room reaches the outer layers of the pier first and only gradually penetrates to the interior parts of the masonry. It has seemed to me, also, that the occurrence is not merely a question of temperature, but rather a case of drying out the parts of the brickwork, from the very damp conditions in the summer to the desiccated condition during the winter months. The lower curve is naturally more sinuous than the upper, but not nearly as much so as would have been expected in comparison with the curve of fig. 14 in § 17. In fact, on comparing these two curves after September 28, the initial maximum on October 1 and the minimum on October 6 correspond. After this the curves diverge and there is no correspondence until the maximum of November 1 is reached. Thus, for instance, on October 19, fig. 14 finds no counterpart in fig. 9. A comparison of the two curves is naturally immensely in favor of the pier, though even the latter is again altogether inadequate for the kind of work contemplated; *i.e.*, work involving variations in α of hundredths of second of arc. Changes of α of half a second are by no means uncommon, and even in the summer changes of 2 seconds within a month would be a small estimate. The curve shows the nature of the difficulties encountered, for instance, in endeavoring to measure the repulsion of two disks in Chapter II.

Between September 17 and September 28 the temperature curve (light line) is drawn on a large scale, between September 28 and November 5 on a smaller scale. With regard to the former it is evident that the sinuosity of the curves is about the same, but that the minima and maxima are not cotemporaneous. Thus, for instance, at a the temperature minimum precedes the inclination minimum. The same is true for the maximum at b, etc. In their details and, in general, quantitatively, the two curves do not coincide in character. Hence, the effect of temperature, if admitted, can at best be indirect; *i.e.*, temperature changes the inclination α by straining or warping the pier.

If we compare the temperature curves between September 28 and November 5 with the inclination curve, there is again a general resemblance. Thus the maxima at f, g, k and the minimum at h occur in both. But there is no detailed resemblance, even when the difference of scale is taken into consideration. Neither is the temperature effect as marked as in the corresponding case of fig. 14, Part I. The temperature maxima tend to precede the inclination maxima, etc. Hence, as before, temperature acts, not upon the pendulum mechanism directly, but rather indirectly through the supports, which become displaced by unequal expansions in the pier and a corresponding tilting from its position.

Finally, the changes of inclination α shown by the aluminum pendulum are quite as marked as those occurring in the corresponding case of the steel pendulum, although the viscosity error of the former is much greater. It does not therefore appear that the effect of changes of magnetic field has produced any error, such as was surmised above in case of the steel pendulum. The latter is, therefore, preferable for work of the present kind.

CHAPTER II.

THE REPULSION OF TWO METALLIC DISKS, NEARLY IN CONTACT.

26. Apparatus.—The apparatus shown in fig. 28 was originally constructed with the expectation of testing the horizontal pendulum for the measurement of the Newtonian constant; or, conversely, to graduate the horizontal pendulum by means of that constant. Here *AB* suggests the parts of a Fraunhofer slide micrometer, capable of moving the slide about 6 cm. and graduated in 0.0001 cm. On this the two brass disks *DD*, (originally) 15 cm. in diameter and about 0.6 cm. thick, are mounted in parallel, rigidly, normally and vertically. To adjust the disks the steel plugs *c* and *c* are provided, fitting radial holes in the plate. They are further held by the semicircular frame *e* and *e*, screwed to the slide below and attached above to the disks by aid of the pairs of screws, *a* and *b*, on opposite sides of the diameter. The screw *a* is sunk into the disk, while *b* presses against its outer surface. As the disks are to be fitted nearly true to the slide and the frame, but slight adjustment at *a* and *b* is needed.

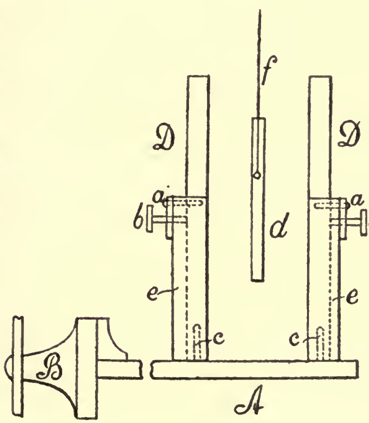


FIG. 28.

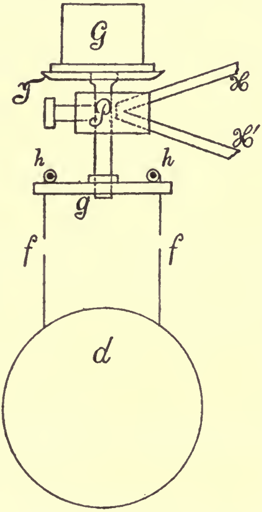


FIG. 29.

The interior of the smaller disk *d*, (originally) about 10 cm. in diameter and 0.6 cm. thick, is suspended vertically by two fine wires *f* from the end of the arm of the horizontal pendulum, just below the grating. The disks *D*, *d*, *D* are coaxial, while *d* is relatively stationary; *D* or *D* may be brought as near to *d* as desirable by aid of the slide micrometer, the other disk being removed at the same time.

The method of attachment of the disk d to the horizontal pendulum is shown on a smaller scale in fig. 29. Here G is the grating, secured by three adjustment screws to the table T, the cylindrical shaft of which is grasped on a clamp (open form) of the horizontal pendulum P. To the bottom of the shaft in question, a cross-piece *hgh* is screwed and fastened with a lock-nut. The two fibers *ff* which support the disk d are wound above around the pulley screws *hh* and thus adequate vertical adjustment of disk d is available.

The slide micrometer is attached to the pier by a firm horizontal rail capable of adjustment forward and rearward. A strong clamp attaches the base of the slide micrometer to this rail, so that the whole instrument may also be adjusted to the right or left, roughly. The fine adjustment is completed on the slide micrometer itself.

Finally a case is provided covering the disks D and d and part of the micrometer, so that only the drumhead and scale projects. The apparatus was found to work satisfactorily. It is quite possible to reject the water damper at the end of the horizontal pendulum, above, and to rely solely on the effective air damping produced, when the disk d is very close to D or D'. In fact, the tin cover, in this case, was all but superfluous. D could be shifted from end to end of the course, without materially interfering with the visibility of the ellipses in the spectrum of the interferometer. The real interferences unfavorable to the gravitational measurement were incidental, due either to the change in inclination of the pier, or to changes in the magnetic field (inasmuch as the pendulum was preliminarily constructed of steel tubing), or to the causes discussed in this chapter; for what was found was not an attraction at all, but a *repulsion*, much larger in absolute value than the attraction anticipated.

27. Equations.—The chief equations to be used in the present work have already been given above. It is merely necessary to add those which bear upon the sensitiveness of the method. Since the disk of mass m is added, at the mean distance R, to the mass of the pendulum M, the force at R from the axis is now

$$F'_R = F_R\left(1+\frac{m}{M}\frac{R}{h}\right) \tag{1}$$

The gravitational attraction f' of the disks necessarily involves spherical harmonics, but may be written temporarily as

$$f' = \gamma m m'/f(d) \tag{2}$$

where m' is the mass of the stationary disk at a mean distance d from m. Equating these forces and inserting the value of F_R, the equation for ΔN, the displacement at the micrometer, becomes

$$\Delta N = \frac{\gamma m' \frac{m}{M} 2R^2}{f(d) g\varphi h\left(1+\frac{m}{M}\frac{R}{h}\right)} \tag{3}$$

In the first place, therefore,

$$\Delta N \propto \frac{m/M}{1+\frac{m}{M}\frac{R}{h}}$$

so that the lightest available pendulum and the heaviest admissible disk is to be selected, although the increase of sensitiveness is not quite proportional to m/M, but diminishes as $(1+mR/Mh)^{-2}$. This procedure, even when the float is used, is relatively inefficient and the value of ΔN can probably not be increased more than twice the above value (a difference of $\Delta N = 2 \times 0.0012$ for the two extreme positions of the disk) by this means.

28. Equations for the vertical pendulum.—A final word may be added with regard to the inclination α. This can be detected with such precision that a method based upon it deserves consideration. The apparatus in this case would take the form of fig. 30, where $ABCD$ is the iron framework of the heavy, long, vertical pendulum, with the massive bob at D and knife-edges and tablets at e, so that the pendulum is capable of swinging normally to the plane of the diagram. The horizontal pendulum is attached by two pivots, a and b, to the central rod CD of the vertical pendulum. It is to swing clear of it and to be in equilibrium in a parallel plane. The deflection of the horizontal pendulum is also normal to the plane of the diagram, and it measures the change of α of CD, as above, G being the grating, h the center of gravity.

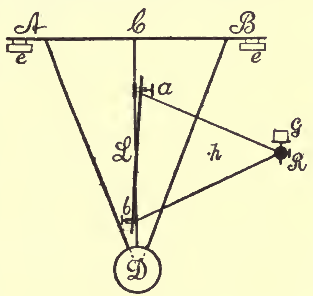


FIG. 30.

When gravitational attraction is to be observed, the bob D is one of the attracting bodies and of mass m', whereas the attracting mass m, with its center on the same level, is placed in front of or behind the plane of the diagram.

If the mass m' at the end of the vertical pendulum is at the distance L from the horizontal axis, and the mass M' of the remainder of the pendulum virtually at a distance H (center of gravity) from the axis,

(5) $$M'gH\alpha + mgL\alpha = L\gamma mm'/d^2$$

where d is the mean distance apart of m and m'. Hence

(6) $$\Delta N' = \gamma m 2R/\varphi d^2 g(1+M'H/m'L)$$

If m' is massive, so that $M'H/m'L = 1$ may be assumed; if the bodies m and m' are equal spheres of radius r all but in contact,

$$d = 2r \text{ and } \Delta N' = \gamma\pi\rho Rd/6\varphi g$$

Thus if $d = 10$ cm., $\rho = 10$, with the other magnitudes as in the above interferometer, $\Delta N' = 10^{-5} \times 7.5$ cm., which increases but as the first power of the diameter of the spheres. Hence, in spite of the precision of α measurement, the method would not be available for the determination of γ.

29. Observations with small plates.—The first experiments were made merely for the purpose of testing the method, using the same heavy horizontal pendulum as in the preceding section. There are two or three objections to this pendulum for the present purposes, of which the first is its weight M; the second is the water damper, which introduces inevitable discrepancies, due to such capillary forces as result from surface viscosity. The third objection is due to the fact that the pendulum is made of light steel tubing and points in the north-south direction. These tubes become weak magnets in the earth's field, and the angle θ may change with the variations of this field. Finally the inclination α of the axis of the pendulum, due to terrestrial causes, is itself to be considered; this can only be eliminated if the time of observation is reduced.

The two attracting plates of rolled brass were each 6 inches in diameter and 0.25 inch thick, weighing $m' = 1{,}035$ grams. The attracted disk d attached to the horizontal pendulum was 4 inches in diameter and 0.125 inch thick, weighing 227 grams. The distance between the large plates was 2.5 cm. on the micrometer, this being about the limit of the micrometer screw and sufficient for the diminution of the attraction in question to negligible values. The difference of ΔN for the two extreme positions of the disks was estimated above as 0.0024 cm., or 5 drum-parts. It should have been easily detected, if not masked by the incidental disturbances referred to.

The five series of observations are given in the curves, figs. 31A, 31B, 31C, 32A and 32B. They show both the release of the suspended disks from contact with the disk fixed on the micrometer, and the differential effect of the fixed disks on opposite sides of the suspended disk, but near it.

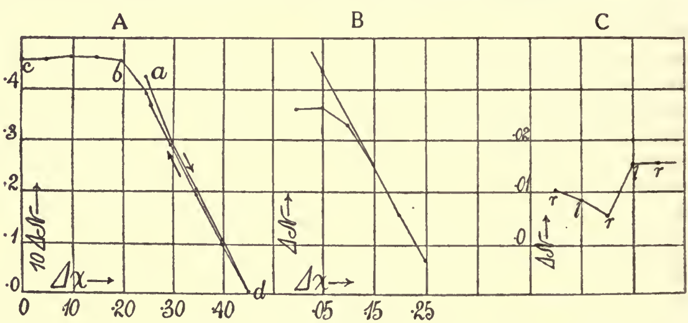


FIG. 31.

In fig. 31A, the abscissas are the successive excursions Δx of the micrometer bearing the fixed plates, the ordinates are the corresponding excursions ΔN of the suspended plate. Beginning at a, the two plates are nearly in contact, and this contact is made more definite in the direction $+x$. Hence in the curve from a to d to b, as shown by the arrows, $2\Delta N = \Delta x$, as it should be. After passing b toward c the suspended plate is released, but released in such a way as to suggest repulsion at b, whereas the other four points nearer c

in their downward slope toward the left would be compatible with gravitation. Release should take place at the intersection of the two lines. Results of the same kind are shown in fig. 31B, where the apparent repulsion is very definite. It is probable that in both cases the discrepancies observed are distorted by capillary forces, surface viscosity at the water damper, and by the inclination difficulties. This is borne out by fig. 31C, in which the fixed disks were alternately placed all but in contact with the suspended disk. The curve should have been zigzag, with the oscillations equal and in opposite directions; but it is quite irregular, due to extraneous causes. R and L indicate whether the fixed disk is on the right or left side of the suspended disk.

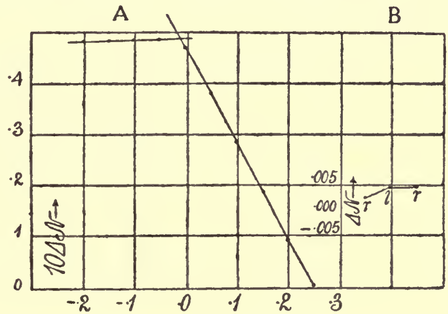


FIG. 32.

The water damper was now removed and the work repeated, relying on the air-damping at the disks only. No difficulty was experienced in obtaining the interferences; but the results fig. 32A show no evidence whatever of attraction. Similarly in the alternations of fig. 32B, the curve which should have been zigzag shows no regularity. Here again foreign disturbances have masked the effect sought, although the displacements themselves were apparently definite and satisfactory. It is therefore necessary to replace the disks by a larger set, as is done in the next section.

30. Observations. Plates of larger area.—The brass plates were now replaced by a set larger in area but thinner, this being in the direction of the improvement of method indicated. The same unnecessarily heavy steel pendulum had, however, to be used, so that $M=1,250$ grams, $h=80$ cm., $R=111.3$ cm., $\varphi=0.01081$ radian, $F_R=42.9\ \Delta N$. The new brass plates were identical in size, the mass being $m=468$ grams each, the diameter $2r=20.3$ cm., and the thickness 0.17 cm. In place of gravitational attraction an apparent *repulsion*, equivalent on the average to 0.0338 cm., or about 68 drum-parts, was observed.

The observations are given in table 1 and in figs. 33A, 33B, 33C, the arrows showing the direction of successive observations. The abscissas denote the positions Δx of the attracting "fixed" plate on the micrometer, the ordinates the corresponding value of the displacement ΔN of the plate suspended from the horizontal pendulum, read off on the micrometer of the interferometer. The plan was to begin with the plates more than in contact, so that the movable disk is carried by the fixed disks until released. In fig. 33A, at *b*, the plates adhere until at *a* a release or fall suddenly takes place, the plates being now over 1 mm. apart. The figure shows that release should have occurred at the position 2.09 cm. In fig. 33B, corresponding to the other ("small") side of the Fraunhofer carriage, the plate is released at the position 0.51 cm. and there is no adhesion. In fig. 33C, on the original ("large") side, the plate is passed into cohesion and then released with a smaller fall at *a*.

TABLE 1.—Large brass disks, not in metallic contact. Steel horizontal pendulum. $m=468$ grams; $r=10.2$ cm.; $t=0.17$ cm.; $\varphi=0.0108$; $M=1{,}250$ grams; $h=80$ cm.; $R=111.3$ cm.; $F=43\times\Delta N$; $F'_B=65.2\Delta N$.

Fixed plate at Δx	Movable plate at $10^4\Delta N$	Fixed plate at Δx	Movable plate at $10^4\Delta N$
Fig. 33A. 2.15 cm.	− 955	Loosening. 2.05 cm.	+650
.10	+ 37	2.05	185
.05	1030	2.00	295
.00	2015	2.05	215
1.95	2875		
.90	283		
Fig. 33B. 0.45	2493	Fig. 33D. 2.00	*302
.40	3493	{.60	580
.50	1605	{.65	517
.55	625	2.00	187
.60	495	.65	565
.65	495	2.00	150
		.65	535
Fig. 33C. 0.65	495	2.00	177
1.90	285	.65	550
1.95	320		
2.00	320		
2.05	+ 175		
2.10	− 285		

* Mean values.

Thus the positions 2.00 and 0.65 are guaranteed as free, the space between the reacting plates being over 1 mm., as compared with the distance 1.4 cm. between the fixed plates. The effects of alternately approaching the opposed fixed plates to the movable disk are shown in fig. 33D. They are quite definite, larger in order of value than would be anticipated and constitute repulsions instead of attractions. In fact, figs. 33A and 33B show that in case of attraction or of cohesion, ΔN should be too large on the "large" side, and too small on the "small" side of the stationary disk. In fig. 33D the reverse is the case.

To explain this repulsion a number of facts have to be taken into account. Both the fixed disks are separate metallic systems, but ultimately anchored into the pier with iron bolts, so that a volta contact force, iron-brass, would be inevitable. The disks are thus carrying charges, depending on the nature of the anchorage in the pier, whether this is moist or quite dry. It seems probable, as will be shown below, that these small potentials are negligible. Again, with the small forces per square centimeter of area in question, the viscosity of air is sufficient to necessitate the lapse of considerable time before a position of equilibrium is assured, even with the disks a millimeter or more apart. In the above experiment sufficient time was allowed until the motion became vibratory, after which the reading was taken; but it is difficult to assert that, even after indefinite waiting, further subsidence would not have taken place. Finally the flexure or tipping of the pier, where long intervals of time are in question, can not be eliminated. There would inevitably be some error on this account. It seems improbable, therefore, that the actual gravitational attraction of metallic disks will be determinable, while a non-metallic system is liable to introduce even greater errors.

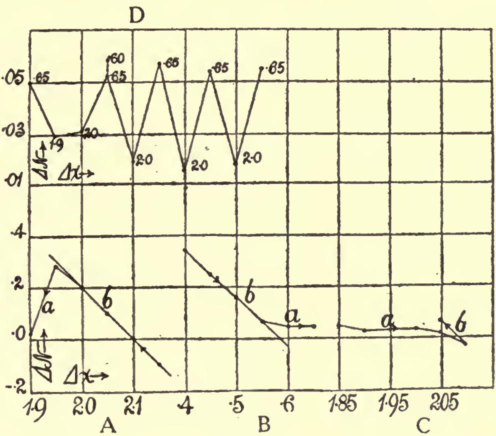


FIG. 33.

31. The same, continued. Metallic contact.—The next advance consisted in placing the disks in electrical (metallic) contact, which was easily done by joining the pivots of the horizontal pendulum with the slide of the micrometer bearing the fixed disks by a copper wire. Moreover, since the position of equilibrium is gradually reached in the lapse of minutes, the time of the observations is taken in minutes. These results are given in table 2, and are inscribed in figs. 34A and 34B. The figures on the curve show the series in question and the plate ("large" or "small" side of the plate micrometer), which is actively repelling. In fig. 34A the alternations are found after long waiting; in fig. 34B, however, in time series. When equilibrium is reached, and this is always relatively quickly, the disk oscillates due to incidental causes. It makes no difference from which side the position of equilibrium is approached (series 2, 8, 15). The presence of radium on the plates has no effect, other than the mechanical disturbance given by placing it there; the same position of equilibrium again results. When the disks are jolted by contact (case between series 7 and 8), the equilibrium position may be tem-

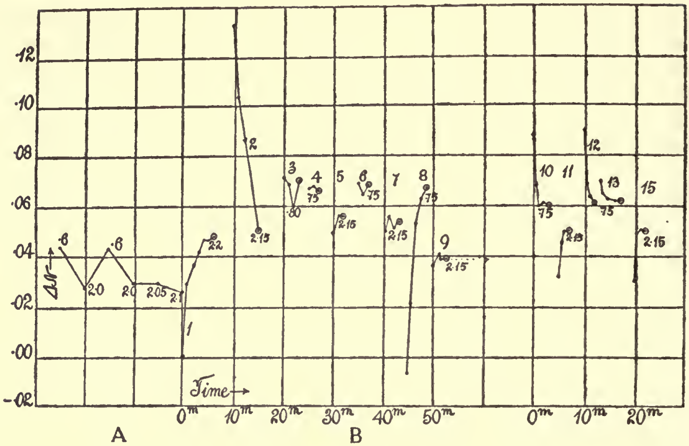


FIG. 34.

TABLE 2.—Large brass disks in metallic contact. Constants as in table 1.

Δx	$10^3\Delta N$	Time.
cm.	*cm.*	*min.*
0.65	44	.....
2.00	28	.....
.65	43	.....
2.00	29	.....
2.05	29	.....
2.10	26	.....
2.20 I	0	0
	29	2
	36	4
	41	6
	46	8
	46	10
	47	12
2.15 II	132	0
	103	2
	† .50	10
0.80 III	71	0
	68	2
	? 59	4
	70	6
0.75 IV	66	0
	67	2
	65	4
2.15 V	48	0
	56	2
	55	4

Δx	$10^3\Delta N$	Time.
cm.	*cm.*	*min.*
0.75 VI	68	0
	64	2
	68	4
2.15 VII	28	0
	49	1
	55	2
	*50	4
	53	5
	53	6
0.65	144	
0.75 VIII	− 7	0
	21	1
	†53	3
	*62	5
	66	7
2.15 IX	36	0
	50	2
	*48	3
	48	4
	48	5
	48	53
0.75 X	88	0
	68	1
	60	2
	61	4
	60	6

Δx	$10^3\Delta N$	Time.
cm.	*cm.*	*min.*
2.15 XI	31	0
	46	1
	49	2
	45	4
0.75 XII	90	0
	68	1
	63	2
	60	4
0.75 XIII Radium on	69	6
	65	7
	62	8
	62	9
0.75 XIV Radium off	62	12
	62	14
2.15 XV	30	0
	48	1
	50	2
	52	4
2.15 Radium on	59	..
	54	..

* Vibrating. † Fall.

porarily disturbed (series 8). Apart from flexure of the pier, long waiting (50 minutes in series 9) does not further change the position of equilibrium, if the slight swinging is taken into account. The cause of the gradual motion may reside in the viscosity of air, as indicated in the next paragraph.

If we compare the results of fig. 33D (system not in metallic contact) with the present (system metallically connected) the results appear as follows:

System not in metallic contact	System metallically connected.
$2\Delta N=0.021$	$2\Delta N=0.016$
22	14
38	.015
38	12
37	13
	10
	11
Mean $2\Delta N=0.038$	Mean $2\Delta N=0.013$

The disks were usually about a millimeter apart. Metallic contact has thus apparently made the repulsion smaller; but it is not certain that the distance apart of the plates is quite identical. Moreover, data obtained at different times vary considerably. In the present case the repulsion observed for the disks 20 cm. in diameter is $2F_R=65.2\Delta N=65.2\times 0.013=0.85$ dyne, at about $d=1$ mm. of air-space.

32. Retardation due to viscosity of air.—It will next be necessary to examine the above suggestion, that the very gradual approach of the suspended disk to its position of equilibrium may be due to the viscosity of the interposed film of air, in view of the small forces and small displacements involved. The case may perhaps be treated in terms of Poiseuille's law, assuming that the flow is from the center of the two nearly contiguous parallel disks radially toward the circumference. Let y_0 be the initial distance apart of the disks, and the time $t=0$ second, measured from the fixed toward the movable disk. Let y' be the final position of equilibrium of the movable disk, so that its excursion is y_0-y'. Let a small impulsive force P act normally on the outside of the movable disk, by which it is put into the position y. The pressure generated will cause a flow radially outward, and if p is the pressure in the fluid at a distance r from the center, Poiseuille's law may be written

$$(1)\qquad -\frac{dV}{dt}=-\dot{V}=\frac{(2\pi r\cdot y)^2}{8\pi\eta}\frac{dp}{dr}$$

for the flow through a ring whose section is $y.dr$, if η is the viscosity of the gas and $\dot{V}$ the volume of fluid crossing per second. If the flow is steady, so that $\partial p/\partial t=0$ for all distances from the center, and if the liquid is virtually incompressible, *i.e.*, $\dot{V}$ independent of r, the problem may be solved without difficulty. Neither of these conditions is quite true. The second, however, inasmuch as the average pressure increment is exceedingly small relative to atmospheric pressure, may be admitted. Suppose, therefore, that at any time, y and $\dot{V}$ are constant relatively to r, and integrate the equation. Then the pressure excess at r is

$$p=\frac{2\eta\dot{V}}{\pi y^2}\left(\frac{1}{r}-\frac{1}{R}\right) \tag{2}$$

If R is the radius of the disks, $p=0$ at $r=R$, and the equation may be considered to hold short of $r=0$. Thus the thrust P becomes

$$P=\int_0^R 2\pi r dr\cdot p=2\eta R\dot{V}/y^2 \tag{3}$$

But

$$\dot{V}=-\frac{d}{dt}(\pi R^2\cdot y)=-\pi R^2 dy/dt \tag{4}$$

whence

$$-Pdt=2\pi\eta R^3 dy/y^2 \tag{5}$$

or, on second integration,

$$Pt=2\pi\eta R^3\left(\frac{1}{y}-\frac{1}{y_0}\right) \tag{6}$$

But for the horizontal pendulum the force P is proportional to $y-y'$, which may be written

$$P=P_0(y-y') \tag{7}$$

so that (5) becomes

$$P_0 dt=2\pi\eta R^3 dy/y^2(y-y') \tag{8}$$

This, on again integrating, becomes finally

$$t=\frac{2\pi\eta R^3}{P_0 y'^3}\left\{-\left(\frac{y'}{y}-\frac{y'}{y_0}\right)-\log\frac{1-y'/y}{1-y'/y_0}\right\} \tag{9}$$

natural logarithms being in question. Since $y-y'=\Delta N/2$, equation (7) corresponds in case of the large disks to $P=F_R'=2\times 65.2\Delta N/2$. Hence, $P_0=2\times 65.2$. Furthermore, in case of this apparatus and in the present experiments, the following data may be entered: $P_0=2\times 65.2$; $y'=1/15$ or $y'/y_0=2/3$; $R=10.2$ cm.; $y_0=0.10$ cm.; $\eta=190\times 10^{-6}$, so that roughly

$$t=2\cdot 19\left(-\left(\frac{1}{15y}-\frac{2}{3}\right)-\frac{2}{3}\log\left(3-\frac{1}{5y}\right)\right) \tag{10}$$

Table 3 contains some corresponding values of y and P computed in this way.

TABLE 3.—Motion of movable brass disk retarded by viscosity of air film.

$10^4\times$ $y=\Delta N/2$	t	$10^4\times$ $y=\Delta N/2$	t
cm.	*sec.*	*cm.*	*sec.*
1000	0.00	680	5.48
900	.39	670	8.47
850	.70	668	10.47
800	1.16	666.8	19.41
750	1.93	$y'=1/15$	∞
700	3.68		

These values are reproduced in fig. 35, which with table 3 shows that the position of equilibrium is reached to $\Delta N = 10^{-4}$, the smallest quantity easily measurable, in about 25 seconds.

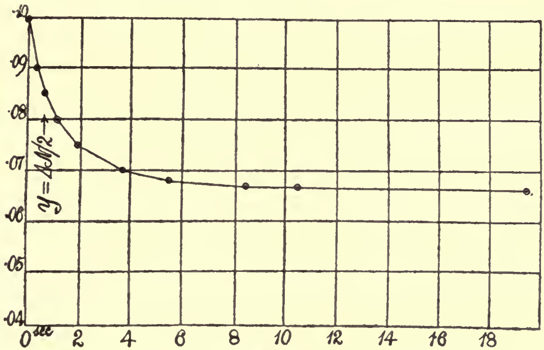


FIG. 35.

The fluid has been treated as incompressible. If this is not done, the results apparently become unavailable. A further step may, however, be made: Poiseuille's equation (1) if the condition $\dot{V}P = \dot{V}_0 P_0$ is introduced, leads on integration to the form

$$p^2 - p^2_0 = \frac{4\eta \dot{V}_0 p_0}{\pi y^2}\left(\frac{1}{r} - \frac{1}{R}\right) \tag{11}$$

where P is the pressure and $\dot{V}_0$ the volume issuing at the edge, per second at the normal pressure p_0. In endeavoring to use (11) directly, I have not succeeded in producing a practical form of equation.

Equation (9) may be put in a different form suitable for computing in the ultimate times of very close approach to equilibrium. For this purpose, let

$$y'/y_0 = a \text{ and } y - y' = b$$

where b is to be very small, so that $y = ay_0 + b$. Equation (9) then reduces nearly to

$$t = \frac{2\pi\eta R^3}{P_0 a^2 y^2_0}\left\{ -\left(1 - a - \frac{b}{ay_0}\right) - \log\frac{b/ay_0}{1-a}\right\}$$

Usually b/ay_0 may be neglected compared with $1-a$. Thus if $b = 10^{-4}$ cm., $t = 11.1$ sec., with the other constants as above, $y_0 = 0.1$ cm.; $a = 2/3$. For the same case, $b = 10^{-4}$ cm., if $y_0 = 0.05$ cm., $a = 2/3$, $t = 38.3$ sec., are needed to approach within 10^{-4} cm. of the position of equilibrium, etc. In case of repulsion, $a > 1$ and b is negative. Thus for $a = 3/2$ cm., $b = 10^{-4}$ cm., $y_0 = 1/15$ cm., $t = 6.53$ sec. For $y_0 = 2/45$ cm., $y' = 1/15$ cm., $t = 13.8$ sec., etc. The intervals so computed are small as compared with the times actually observed, where many minutes have to elapse before equilibrium is obtained. It seems difficult to interpret this excess by supposing that the method is inadequate; for the effect of gravitational attraction between the disks, which has been ignored, would be a virtual increase of P_0. Since, on the average, pressure excess, $p-p_0$, is very small as compared with p_0 in the actual case, equation (11), seeing that $p^2-p^2_0=2p_0\,(p-p_0)$ becomes

$$p-p_0=\frac{2\eta V_0}{\pi y^2}\left(\frac{1}{r}-\frac{1}{R}\right) \tag{12}$$

which is identical in interpretation with equation (2) above, where $p_0=0$ and therefore leads to the same conclusion.

33. Observations, continued. Presence and absence of electrical contact.— Notwithstanding the improbability of electrical effects, it was thought necessary to test the case directly. Accordingly, in table 4 and fig. 36, series 1 to 8, experiments are recorded with the plates not in metallic contact, series 1 to 5, and with the plates in metallic contact, series 6 to 8, respectively. The behavior in both cases is virtually the same, when the shift of zero is taken into account. Observations are plotted in time series, with the last observation marked by a circle, and they are in each case continued until the motion of the plate is retrograde, whereupon the real oscillation of the plate begins. To throw further light on the subject, a Leclanché cell was introduced in series 9 and 10 and removed in series 11. The lighting circuit of the room was placed more remote in series 12 and the system earthed in series 13 and 14 for both fixed disks.

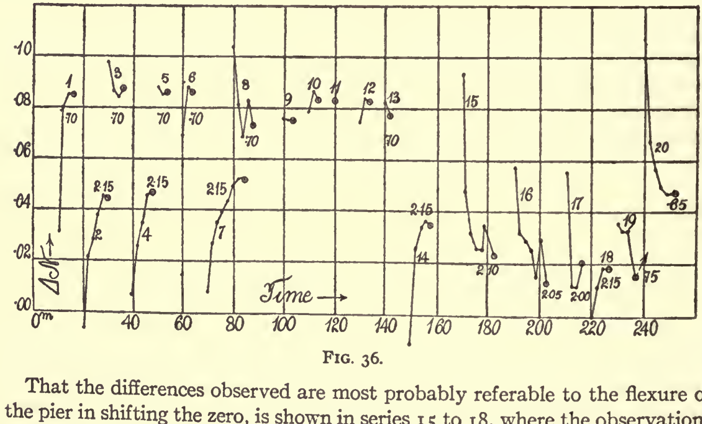


FIG. 36.

That the differences observed are most probably referable to the flexure of the pier in shifting the zero, is shown in series 15 to 18, where the observations are made on one side only, with the distance between disks gradually increasing, as the fixed plate moves from position 2.15 to position 2.00, *i.e.*, 0.15 cm. The interval of observation was 48 minutes; but the interval and final reading

TABLE 4.—Displacement of brass disk with electrical contact and without. Constants as in table 1.

Δx	$10^3\Delta N$	Time
cm.	*cm.*	*min.*
0.65	93	46
0.70 I	32	48
	80	50
	85	52
	85	54
2.15 II	− 6	57
	+22	59
	28	61
	38	63
	45	65
	44	67
0.70 III	98	10
	87	12
	84	14
	87	16
2.15 IV	7	19
	26	21
	35	23
	46	25
	47	27
0.70 V	88	30
	85	32
	86	34
0.70 VI	69	39
	88	41
	86	43

Δx	$10^3\Delta N$	Time
cm.	*cm.*	*min.*
2.15 VII	8	46
	27	48
	35	50
	39	52
	43	54
	49	56
	52	58
	52	60
	52	62
0.70 VIII	104	4
	81	6
	69	8
	83	10
	*73	12
0.70 IX Battery on	76	16
	75	18
	75	20
0.70 X Poles exchanged	79	22
	87	24
	83	26
0.70 Battery off	83	28

Δx	$10^3\Delta N$	Time
cm.	*cm.*	*min.*
0.70 XI	74	31
	84	33
	83	35
0.70 XII Earthed	83	44
	77	46
2.15 XIV	−12	50
	+25	52
	34	54
	36	56
	35	58
2.15 2.10 XV	45	16
	93	18
	48	19
	32	21
	26	23
	26	25
	34	27
	23	31
2.05 XVI	57	33
	32	35
	29	37
	25	39
	15	41
	29	43
	12	45

Δx	$10^3\Delta N$	Time
cm.	*cm.*	*min.*
2.00 XVII	55	46
	11	48
	11	50
	20	52
2.15 XVIII	0	58
	11	60
	18	62
	18	64

Δx	$10^3\Delta N$	Time
cm.	*cm.*	*min.*
0.75 XIX	36	7
	33	9
	33	10
	24	12
	15	14
0.65 XX	100	17
	67	19
	57	21
	50	23
	47	25
	47	27
	47	29

*Next day, 71.

at 2.15 cm. differ in like degree, so that the apparent attraction indicated is merely the result of the shift of the position of equilibrium of the horizontal pendulum. Series 19 and 20 contain similar observations on the other side.

34. Observations, continued. Change of distance apart.—In the following work table 5 and figs. 37, 38, 39A, 39B, the attempt is again made to vary the distances between fixed and movable plates, successively, but to determine the micrometer position of contact of the plates by actually pushing them together with a weak spring. Observations are made on the "larger" side in series 1 to 8, the distance apart increasing from position 2.15, where it is 0.045/2 cm., to position 2.05, where it is 0.247/2 cm., *i.e.*, the total displacement being about 0.1 cm. The observations are given in time series in fig. 37 and are not interrupted, until the motion of the pendulum is retrograde. The doubly inflected curve, series 3, 4, 5, etc., is well shown; *i.e.*, the plates after immediate contact separate very slowly, whereupon the speed of separation reaches a maximum, to decrease to zero again when the pendulum regains its position of equilibrium. It is probable that in series 1 and 2 the

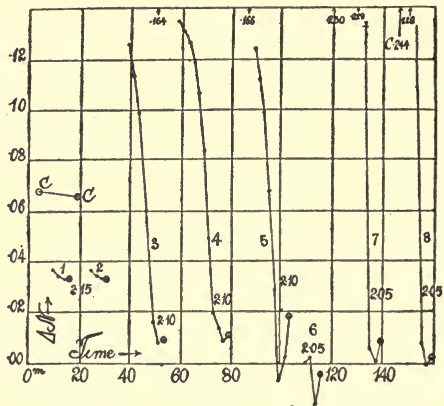


FIG. 37.

pendulum has not separated as far as its position of equilibrium, the motion here being excessively slow. The whole of the motion in series 1 to 8 may be followed by moving the micrometer as the ellipses pass through the spectrum.

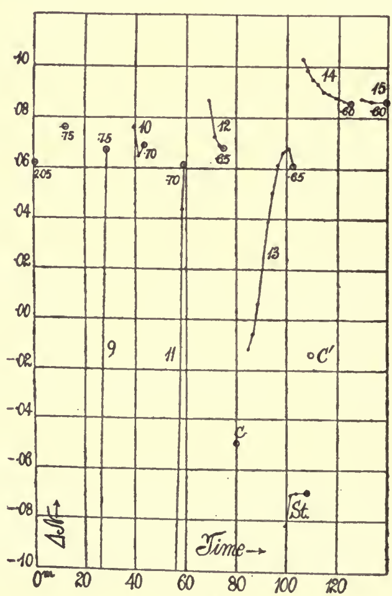


FIG. 38.

If the mean results be taken from the figure, the following data appear:

Position of fixed plate	2.05 cm.	2.10 cm.	2.15 cm.
ΔN. Plates in contact	.247	.165	.067
ΔN. Suspended plate free	.000	.010	.022
ΔN. Distance apart of movable and fixed plate	.247	.165	.045

These results are shown in fig. 39A. They are such as to indicate an apparent attraction of the fixed and movable disks, increasing as their distance apart diminishes. Observations were now made on the other side to see whether the results would be corroborated, or whether in the two hours of observation the shift of the position of equilibrium of the pendulum was so

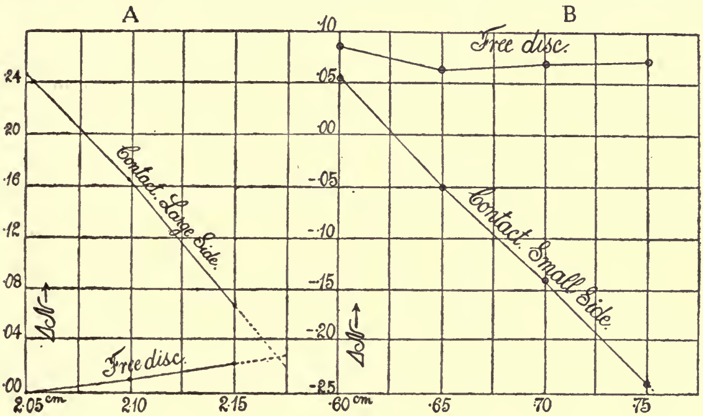


FIG. 39.

great as to obscure the true conditions completely. The new results for the "small" side are given in fig. 38. They exhibit the familiar inflected curves whenever the pendulum separates from the position of contact, which are observed until the pendulum begins to swing to and fro. In series 14 and 15 the movable disk was probably not quite free. The mean results may be estimated as

Position of fixed plates	0.60 cm.	0.65 cm.	0.70 cm.	0.75 cm.
ΔN. Plates in contact	.055	−.050	−.142	−.240
ΔN. Suspended plate free	.086	.064	.070	.071
ΔN. Distance apart of fixed and movable plate	.031	.114	.212	.311

When the plates are 0.150 cm. apart (0.75), the air-damping is insufficient. The ellipses move to and fro in the spectrum. These results are given in fig. 39B, and, apart from the observation at 0.60, they contain no evidence of any gravitational effect beyond the limits of error. The time needed was 2 hours, within which the shift of the position of equilibrium of the horizontal pendulum is not guaranteed.

It does not seem possible, therefore, to obtain any definite results from methods which necessarily consume as much time as the present.

TABLE 5.—Brass disks alternately in contact and free on one side. Constants as in table 1.

Remarks.	Δx	$10^3 \times \Delta N$	Time.
	cm.	*cm.*	*min.*
Free........	2.15	21	..
Contact....	2.15	68	..
Free........	2.15	37	53
I		34	55
		33	57
		33	59
Contact....	2.15	66	..
Free........	2.15	36	3
II		34	5
		33	7
		33	9
Free........	2.10	126	12
III		113	14
		99	16
		60	19
		16	22
		8	24
		9	26
Contact....	2.10	164	..
Free........	2.10	135	31
IV		131	33
		127	35
		119	37
		106	39
		84	41
		49	43
		20	45
		13	47
		9	49
		10	51
Contact....	2.10	166	..
Free........	2.10	124	56
V		112	58
		99	60
		78	62
		29	64
		− 8	66
		+ 2	68
		8	70
Free........	2.10	12	3 hours later
Free........	2.05	0	42
VI		*+ 2	44
		− 17	46
		− 4	48
Contact....	2.05	250	..
Free........	2.05	228	54
VII		208	56
		135	58
		*5	60
		0	62
		8	64

Remarks.	Δx	$10^3 \Delta N$	Time.
	cm.	*cm.*	*min.*
Contact....	2.05	244	..
Free........	2.05	226	8
VIII		205	10
		109	12
		†7	14
		− 1	16
		+ 2	18
The other side, next day.			
Contact....	0.75	−240	..
Free........	.75	−200	1
IX		− 7	3
		+ 67	5
Free........	0.70	76	8
		65	10
		68	12
Contact....	0.70	−142	..
Free........	.70	−215	20
XI		− 67	22
		+ 44	24
		+ 61	26
Free........	0.65	87	29
XII		73	31
		69	33
		68	35
Contact....	.65	− 50	..
Free........	.65	− 11	47
XIII		− 5	49
		+ 6	51
		39	55
		50	57
		62	59
		66	61
		68	63
		61	65
Free........	0.60	123	9
XIV		104	11
		99	13
		95	15
		93	17
		90	19
		89	21
		88	24
		87	27
		86	30
		86	44
Contact....	0.60	55	..
Free........	.60	87	51
XIV†		86	55
		86	67

*Swinging. †Total time of observation, over 2 hours.

35. Observations. Long periods and inversion.—The unsatisfactory results obtained in the last paragraph induced me to give a final trial to the original method of alternating the sign of the repulsion, by moving each of the two fixed plates in turn near the suspended plate. The results are given in table 6 and fig. 40. In series 1, 2, 3 the equilibrium position is approached from two opposed directions and for two positions on the large side. The same is the case in series 5 and 6, while in series 4, 7, 8, 9, similar observations are made on the small side. The mean results are

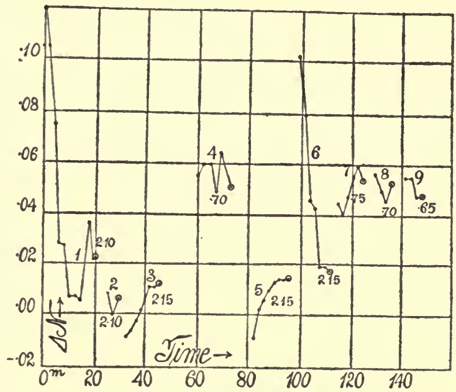


FIG. 40.

Fixed disk at	2.10	2.15	0.70	2.15	0.75	0.70	0.65 cm.
ΔN. Plates in contact	....	.075	−.150	.075	....	.150	....
ΔN. Movable plate free	0.015	.012	.045	.016	.053	.052	.048 cm.
$d=$	....	.031	.097	.030	....	.101	....

Hence there can be no further doubt that the repulsions are real, although their nature has not been made out. When the distance between the disks is larger than a millimeter, the air-damping is insufficient and the free disk unavoidably oscillates, as, for instance, in case of series 1 and 4. The evidence, however, is none the less definite. In series 1 and 3, 5 and 6, 7 and 9, the equilibrium position is approached from opposite directions (the displacements of the horizontal pendulum are in half centimeters).

In order to obtain some reason for this result, one may dismiss the effect of electrical repulsion at once. Experiments, moreover, are to be made in the next section, but rather for the purpose of corroborating the force equation used. Furthermore, friction at the pivots may be excluded, since the pendulum is usually in motion, swinging about its position of equilibrium, so that friction would have operated both ways. There remains the possibility of an excess of pressure in the film of air within a metallic fissure as compared with the surrounding air. To obtain some quantitative data, since $F'_R = 65.2\Delta N$, one may note that the average values of $2\Delta N$ were roughly as follows:

Table 6	$2\Delta N = 0.040$ cm.
Tables 1, 2	$2\Delta N = 0.013$, 0.022, 0.038 cm.
Table 4	$2\Delta N = 0.040$ cm.

Hence

Maximum, $F'_R = 65.2 \times 0.020 = 1.3$ dynes. Minimum, $F'_R = 0.4$ dyne.

As the area of the disk is 324 sq. cm., the corresponding average pressure observed is therefore here

$$p = 4\times 10^{-3} \text{ dynes/cm.}^2 = 4\times 10^{-9} \text{ atm.} = 3\times 10^{-7} \text{ cm. Hg.}$$

in the maximum case, or about one-third of this in the minimum case. These differences are to be referred to the distance apart of the plates which was not always measured in the earlier experiments.

The maximum of gravitational attraction of the air within the crevice by either disk is $\gamma 2\pi\varsigma$ where γ is the gravitational constant and ς the mass of the disk per square centimeter or $\gamma 8.5\times 1.4$, so that $2\pi\varsigma = \gamma 8.8$ dynes per gram

TABLE 6.—Equilibrium of brass disks after long waiting. Constants as in table 1.

Remarks.	Δx	$10^3\Delta N$	Time.	Remarks.	Δx	$10^3\times\Delta N$	Time.
	cm.	*cm.*	*min.*		*cm.*	*cm.*	*min.*
Free........	2.10	119	59	Free........	2.15	−21	6
I		105	1	V		−8	8
		75	3			+2	10
		28	5			6	12
		27	7			10	14
		7	9			13	16
		7	11			14	18
		6	13			14	20
		36	17			15	22
		22	18				
		22	19	Free........	2.10	102	24
				VI		79	26
From 2.05...	2.10	8	22			45	28
Free........		0	24			42	30
II		6	26			19	32
						19	34
Free........	2.15	−21	28			17	36
III		−8	30				
		−6	32	Free........	0.75	44	39
		−2	34	VII		40	41
		+2	36			47	44
		10	39			54	46
		11	41			59	48
		11	43			53	50
				Free........	0.70	56	51
Contact....	2.15	75	..	VIII		49	53
Free........	.70	55	48			45	55
IV		59	50			52	57
		59	53	Free........	0.65	78	59
		48	55	IX		56	61
		64	57			56	63
		50	60			47	65
Contact....	0.70	−150	..			47	67

of air attracted. If the air-film is 1 mm. thick, the mass per square centimeter is thus $0.1\times 0.0013 = 10^{-4}$ nearly, and hence the pressure increment can not exceed $p = \gamma 8.8\times 10^{-4}$. The smallest observed pressure is thus $1/\gamma$ times in excess of the largest computed values. The forces in the fissure must thus be of a different kind from increased gravitational attraction if the excess of air-pressure in the field is to account for the observed phenomenon.

36. Plates electrically charged.—The endeavor is next to be made to overcome the repulsion of plates by aid of opposite electrical charges placed upon them. The experiments at present will necessarily be somewhat crude, since

TABLE 7.—Effect of electrical potential. Constants as in table 1. 12.5 volts applied.

Remarks.	Δx	$10^3\Delta N$	Time.
	cm.	*cm.*	*min.*
Potential off..	0.75	65	..
+12.5volts on .	.75	..	42
I		34	44
		18	46
Potential off ..	.75	24	48
II		28	50
		31	52
		37	54
		43	56
		47	59
		50	61
−12.5volts on .	.75	32	3
III		21	5
		20	7
		20	9
Potential off ..	.75	25	11
IV		29	13
		32	15
		etc.	..
Contact......	0.75	− 9	..
Potential off ..	.75	+63	..
Potential off ..	.80	*51	15
V		57	17
		50	19
		55	21

Remarks.	Δx	$10^3\Delta N$	Time.
	cm.	*cm.*	*min.*
+12.5volts on .	.80	*32	22
VI		20	24
		11	26
		15	28
Potential off ..	.80	39	29
VII		46	31
		51	33
		57	35
		54	37
−12.5volts on .	.80	*33	38
		20	40
		8	42
		11	..
Contact......	.80	−101	..

Mean results.

Volts.	Δx	$10^3\Delta N$	Contact $10^3\Delta N$	d
0	0.75	64	−9	0.036
±12.5	.75	19	(Dist. .004)	
0	.80	50	−101	.075
±12.5	.80	11	(Dist. .050)	

* Swinging.

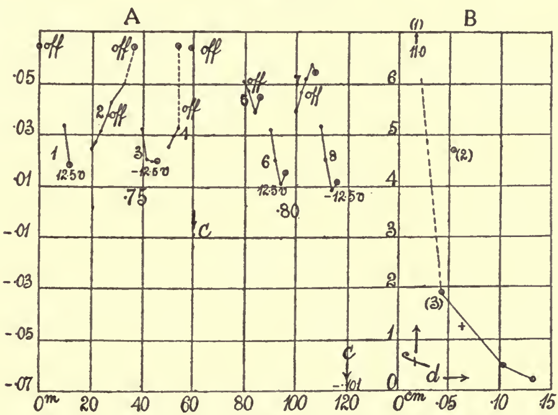


FIG. 41.

the pier is liable to warp during the time in which the observations are made. Nevertheless the corroboration obtained is of great value.

The observations were made in time series, when the plates were close together. For plates farther apart this is not essential, but in the absence of other than the air-damping the suspended plate oscillates, so that mean values have to be taken. It will be necessary to await the summer in order that all observations may be made in a room free from artificial heat.

In table 7 and fig. 41A I have inscribed the first observations from plates close together. In table 8, the summary of all the observations is given. In series 1 to 4, table 7, the plates are probably not far enough apart, though the contact position is still beyond the equilibrium position by about 0.014 cm. It was not necessary to wait for the uncharged position of the plates, as this remained pretty constant long before and after the experiments. In series 5, 6, 7, and 8, the distance apart is increased about 0.042 cm., and there is no danger of actual contact (free space 0.056 cm. when charged), so that actual repulsion is in question.

TABLE 8.—Summary.

d	ΔN	d'	$V_{obs.}$	$V'_{comp.}$	$F_{R\,obs.}$	$F'_{R\,comp.}$	Correction ΔN	$\frac{F_R}{F'_R}$	x	y	f
(1) *cm.*	(2) *cm.*	(3) *cm.*	(4) *volts*	(5) *volts.*	(6) *dynes.*	(7) *dynes.*	(8)	(9)	(10) *cm.*	(11) *cm.*	(12) *dynes.*
0.056	0.039	0.075	12.5	7.43	7.1	2.54	0.110	2.8	0.036	+.020	4.7
.044	.014	.051	6.2	3.56	2.85	.94	.044	3.0	.0146	+.029	1.90
.102	.021	.112	12.5	9.93	2.15	1.37	.033	1.57	.003	+.096	.46
.133	.016	.141	12.5	11.3	1.27	1.04	.019	1.22	.001	+.132	.19
.014	.045	.036	12.5	19.0	114.0	2.93	1.75	39.	.85	−.84	110.

Table 8 shows the essential data of these and subsequent experiments in which the distance d, in centimeters, between the plates is gradually increased. ΔN shows the difference of displacement (in centimeters) observed, when plates were respectively charged and uncharged. ΔN thus measures the displacement of the suspended plate in half centimeters. From ΔN the approximate electric force (ignoring the repulsion of plates) may be computed as above, $F'_R = 65.2\Delta N$. This is given under F'_R computed, in the seventh column.

Plates were charged by aid of a storage battery to the potential shown under V_{obs} (volts), in column 4. From V and d, the attraction of plates F'_R may be computed, since

$$F'_R = \frac{A}{4\pi}\left(\frac{V}{d}\right)^2$$

$A = \pi R^2$ being the area of each plate. The results are given in column 6 and the ratio F/F' in column 9.

Furthermore, the potential V' may also be computed from F_R, since

$$V' = \frac{d}{R}\sqrt{8 \times F'_R} = d\sqrt{8 \times 65.2\Delta N}$$

and this is inserted in the fifth column.

From the computed values of F'_R, since F_R is equal to $65.2 \times \Delta N$, the true value of the displacement in the absence of repulsion, $\Delta N'$, may be computed as

$$\frac{F'_R}{65.2} = \Delta N'$$

the result being given in the eighth column in centimeters. It shows the corresponding displacement of the suspended plate in half centimeters.

The two values of ΔN and $\Delta N'$ now give us the displacement due to repulsion in centimeters,

$$x = (\Delta N' - \Delta N)/2$$

as shown in column 10. Again, the distance apart (in centimeters) of the uncharged plates d' is given in the third column, being

$$d' = d + \Delta N/2$$

and found from observation directly. Finally, the residual distance apart of the plates, y, if the suspended plate had taken its true displacement $\Delta N'$ (in the absence of repulsion), is given in the eleventh column, since

$$y = \frac{\Delta N'}{2} - d'$$

In every case, except the first, in which y is negative, the plates when charged at a distance d apart were not under forces sufficient to put them in contact. One must observe, however, that for a distance apart y when $d < y$, the forces would not increase correspondingly. Only in case 5 is $d = y$, nearly. Thus, without repulsion, the disks should have been thrown in contact when charged, in all cases. In the actual presence of repulsion this was not observed, except perhaps in the first.

The substance of these investigations is contained in column 9, where the ratios of F'_R computed electrically and the value of F_R from independent data, *i.e.*, from the given displacement ΔN of the horizontal pendulum, are given. It is seen that the ratio

$$\frac{F'_R}{F_R} = \left(\frac{V'}{V}\right)^2$$

decreases as the charged plates are farther apart (d), until at $d > 0.13$ cm., the ratio is nearly 1; *i.e.*, the repulsion of plates nearly vanishes when their distance apart markedly exceeds 1 mm. Just how large d would have to be in order that $F'_R/F_R = 1$, I did not endeavor to find, since the suspended plate vibrates annoyingly for large distances apart. In other words, definite experiments of this kind must be left for the summer months. The constants of the pendulum should then also be determined. Moreover, in a lighter pendulum, the sensitiveness may be indefinitely increased, particularly when the pendulum is provided with a float, while the error due to the inclination of the pier does not simultaneously increase, an obvious advantage. It seemed wise, therefore, to stop the work for the present at the point of progress reached.

Table 8, however, admits of a number of preliminary estimates of the decrease of repulsion (f) with d, the distance apart of plates, for we may write,

$$f = 65.2 \times 2x = 130x \text{ dynes, nearly.}$$

These values are given in column 12 of table 8 and in fig. 41B, with the exception of the first, which is liable to be anomalous from actual contact. The second observation also seems to be in error for some reason not detected. The others make a compatible series. The forces found in the above work (paragraph 10) lay between 1.3 and 0.4 dynes, for distances of the same order of value, but which were not quite the same in the two positions of the fixed disks. If we take the results in table 6, which are probably the best, $\Delta N = (0.049-0.014)/2 = 0.018$ cm.; $d = (0.031+0.097)/2 = 0.064$ cm.; $f = 65.2 \times 0.018 = 1.17$ dynes, the results of f and d, as shown by the cross in fig. 41B, fit in very well with the present data obtained from electric attraction. The repulsion therefore has throughout been found of the same order of magnitude.

The pressure corresponding to the above thrust f is found (as above) by dividing by the area A of the disks, whence

$$p = f/A = f/324$$

We may then compute the attraction of the disks per gram of air film, at a distance h from the disk, similarly to the ordinary case of the barometric formula,

$$-dp = \rho F(h)dh = \frac{p}{R\tau}F(h)dh \quad \text{or} \quad -R\tau\frac{d \log f}{dh} = F(h)$$

Thus, if one can detect the variation of f with h, the molecular attraction of the disk per gram of air should be discernible.

37. Conclusion.—By the application of displacement interferometry to the deviations of the horizontal pendulum, I find that two parallel rigid plates whose distance apart is of the order of 1 mm. and less repel each other, in air, with a force far in excess of their gravitational attraction. This force increases rapidly (certainly as fast as the inverse square) as the distance of the plates decreases, and *vice versa*, but can be recognized beyond a millimeter of distance. For brass plates 20 cm. in diameter and 1 mm. apart, the repulsion in question is of the order of 0.5 dyne and therefore equivalent to a pressure of 0.0015 dyne-cm. or roughly 10^{-9} atmosphere. It is in excess of any electric repulsion due to the absolute voltaic potential of the disks. The suspended plate reaches its position of equilibrium gradually, the motion progressing at a retarded rate through infinite time, in a way characteristic of the viscosity of the film of air between the plates.

I have estimated the intensity of the force both from the repulsions of a vertical plate suspended from the horizontal pendulum on opposite sides of a fixed parallel identical plate; also by charging pairs of plates to a given difference of potential for a given distance apart. So far as can be seen, the repulsion is caused by the condensation of air on the surface of the plates by molecular and not by gravitational force (which is too small). Hence, the method employed should enable the observer to find the density of the concentration in terms of the distance from the plate and the law of attraction of the plate in terms of distance within the small distances in question. In other words, a method for direct investigation of molecular force is here apparently given.

Correction: The effect of gravity in Chapter II was overestimated and the data have been withdrawn.

To make an estimate of the gravitational attraction between the largest plates, it is sufficient here to consider the attraction of contiguous disks under the approximate form

$$F = 2\gamma\pi\sigma^2 A = 2\gamma\pi\frac{m^2}{A^2}A = 2\gamma\frac{m^2}{r^2}$$

where $\gamma = 6.7 \times 10^{-8}$ is Newton's constant, σ the density, $m = 468$ g. the mass, A the area, $r = 10$ cm. the radius of the disks. We may then write

$$\Delta N = 2\gamma\frac{m}{r^2}\frac{2R^2\dfrac{m}{M}}{g\varphi h(1 + mR/Mh)}$$

and on inserting the value of the variables $\varphi = \cdot 011$, $h = 80$ cm., $M = 1{,}250$ g., $R = 111$ cm.

$$\Delta N = 4 \times 10^{-6},$$

equivalent to a little over one-tenth of a vanishing interference ring. Hence the gravitational attraction could not have been recognized, as it was not.

CHAPTER III.

THE REFRACTION OF LONG GLASS COLUMNS MEASURED BY DISPLACEMENT INTERFEROMETRY.

38. Introductory.—The measurement of indices of refraction and their differences for different colors, in terms of the shift of the ellipses in the spectrum, seemed to give an opportunity for unusual sensitiveness of method when long columns of glass are inserted in one of the interfering beams. But this expectation was not realized in full, as the amount of shifting per unit of displacement of the micrometer mirror decreases with the thickness of the glass, or the length of the column. The measurements made are nevertheless interesting as a test of the availability of the equation

$$N_c/e = \mu \cos R - \frac{\lambda}{\cos R}\,\frac{d\mu}{d\lambda}$$

where N_c is the coördinate of the movable mirror, corresponding to the center of the ellipse at the wave-length λ of the spectrum. R is the angle of refraction, μ the index of refraction of the glass for the same color (the angle of incidence at the grating being I), and e the thickness of the glass column in the direction of the penetrating ray. For most purposes the Cauchy equation may be used for determining $d\mu/d\lambda$.

39. Glass columns.—The column to be tested is placed with its faces normal and symmetrical to one of the component beams, the corresponding mirror having been advanced proportionately to the length of the column, until the ellipses appear. The fine adjustment is then made with the micrometer screws, the displacement ΔN needed to shift the centers of the ellipses from one spectrum line to the next in succession being observed. This is the chief datum of interest in the present paper; for from it the tests in question may be constructed.

The first columns provided were made from thick glass rods, the ends of which had been ground off normally to the axis of the cylinder by Mr. Petitdidier. But in none of the rods prepared was the glass sufficiently homogeneous to admit of its use. In some cases, in fact, the stress or its equivalent was so strong as to make the rod virtually opaque and the polarization figures correspondingly intense. The attempt was made to remove the stress by annealing at low red heat, but this also was quite unsuccessful.

Failing in all other trials, I finally resolved to build up columns from ordinary plates of glass, cemented together with Canada balsam. In this way I obtained adequately clear columns, though the color of the glass is frequently a disagreeable feature. The three columns made had the following dimensions:

	Length.	Breadth.	Height.	No. of plates.	Average thickness of each.
	cm.	*cm.*	*cm.*		*cm.*
Bluish glass	25.4	6	9	33	0.77
Greenish glass	22.87	6	9	23	.99
Greenish glass	7.1675	6	9	10	.72

In order to keep the plates together, the top and bottom of the long column *AB*, fig. 42, were provided with close-fitting strips of metal *mm* and *m'm'*,

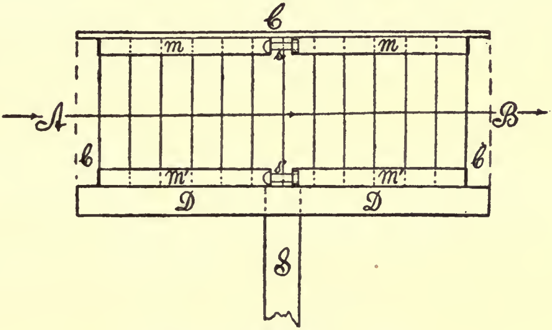


FIG. 42.

tightened by the screws *ss'* on both sides. It was then incased in a wood and metal sheath *CD*, carrying a long stem *S* at right angles to the length of the column, by aid of which it could be clamped in any necessary position relatively to the interferometer.

Over half a year was allowed for the aging of these columns, at the end of which time the balsam was hard (the column having been kept under stress), quite clear, and free from air-bubbles. In fact, the narrow beam *AB* from the collimator, passed through these columns twice, came to an adequately sharp focus in the telescope, and no difficulty in finding or adjusting the interference rings was experienced.

In view of the large glass path the ellipses were necessarily quite small and their motion in response to the micrometer screw sluggish. They were continually in motion, owing to the unavoidable tremors to which the laboratory is subject, indicating of course that the evanescence of rings is still commensurable with the wave-length of light, whereas the sensitiveness of the shift has been reduced in proportion to the length of column. Only in one respect was the behavior peculiar. It was quite common to obtain only the right-hand half or the left-hand half of the set of sharp concentric-ring patterns; *i.e.*, on one side of the vertical line passing through the center of ellipses the interference pattern was quite absent, while it was clear on the other. This seemed to be particularly the case whenever the component ray did not quite retrace its path, *i.e.*, when the paths through the column were separated; but I have been unable to find a reason for the peculiar result.

40. Equations.—In the earlier papers I showed that the coördinate of a movable opaque mirror N_c, varying with the position of the center of the ellipses in the spectrum, in case of the displacement micrometer, could be written

$$N_c = e\left(\mu \cos R - \frac{\lambda}{\cos R}\frac{d\mu}{d\lambda}\right) \tag{1}$$

where e is the thickness of the plate of the grating, or other plane parallel glass body interposed in one of the component beams, R the angle of refraction (the corresponding angle of incidence is I for all colors) for the color λ and the index of refraction μ. If, in addition to the plate of the grating, the column of glass of length E is interposed at normal incidence, $R=0$, but having the same μ for the same λ,

$$N_c = \left\{ (E+e\cos R)\mu - \lambda\left(E+\frac{e}{\cos R}\right)\frac{d\mu}{d\lambda} \right\} \tag{2}$$

For an air-column of length E replacing the glass column for the same color

$$N'_c = \left(E + e\mu\cos R - \lambda\frac{e}{\cos R}\frac{d\mu}{d\lambda}\right) \tag{3}$$

Hence,

$$\Delta N_c = N_c - N'_c = E\left((\mu-1) - \lambda\frac{d\mu}{d\lambda}\right) \tag{4}$$

where the centers of ellipses are brought to the same spectrum line, both for the glass- and for the air-column. Hence, if we write approximately, for brevity,

$$\mu = A + B/\lambda^2;\quad \lambda\frac{d\mu}{d\lambda} = -2B/\lambda^2 \tag{5}$$

$$\Delta N_c = E(\mu-1) + 2EB/\lambda^2 \tag{6}$$

In other words, to measure the index of refraction μ for a given color λ, the correction $-2EB/\lambda^2$ must be added to the corresponding value of ΔN_c, the shift of micrometer, or

$$\mu - 1 = \frac{\Delta N_c - 2EB/\lambda^2}{E} \tag{7}$$

Again, if for the same column the displacement of the micrometer $\delta N_c = N_c - N'_c$, corresponding to different lines of the spectrum, is in question, since

$$\delta N_c = \left\{ (E+e\cos R)(\mu-\mu') - \left(E+\frac{e}{\cos R}\right)\left(\lambda\frac{d\mu}{d\lambda} - \lambda'\frac{d\mu'}{d\lambda'}\right) \right\} \tag{8}$$

the reduced equation becomes

$$\delta N_c = B\left((E+e\cos R) + 2\left(E+\frac{e}{\cos R}\right)\right)\left(\frac{1}{\lambda^2} - \frac{1}{\lambda'^2}\right) \tag{9}$$

so that B may be computed from this equation; or if three terms of the right-hand member be taken and further observation made, B and C may be computed without reference to ΔN_c; *i.e.*, without a knowledge of the displacement due to the length of the column, for any color λ, as a whole. With regard to ΔN_c, I may recall that this quantity is the difference of the air-paths, from the respective points of intersection of the normal with the two faces of the grating at the place where the incident white ray impinges, to the two opaque mirrors.

41. Equations. Sensitiveness in terms of displacement.—To the same extent in which equation (1) applies, the sensitiveness $dN_c/d\lambda$ may now be computed, since

$$dR/d\mu = -\frac{\tan R}{\mu}\frac{d\mu}{d\lambda} \tag{11}$$

and

$$d\lambda/d\theta = -D\cos\theta \tag{12}$$

where D is the grating constant and θ the angle of diffraction for the color λ. Performing the operations and reducing,

$$-\frac{d\lambda}{dN_e} = \frac{\lambda e}{\cos R}\left(\frac{\tan^2 R}{\mu}\left(\frac{d\mu}{d\lambda}\right)^2 + \frac{d^2\mu}{d\lambda^2}\right) \tag{13}$$

or, by inserting equation (5),

$$-\frac{d\lambda}{dN_e} = \frac{\lambda^5}{2eB}\,\frac{\mu\cos R}{2B\tan^2 R + 3\mu\lambda^2} \tag{14}$$

If $R = 0$ (normal incidence),

$$-d\lambda/\lambda^3 = dN_o/6eB \tag{15}$$

or

$$N_e - N_e^\infty = 12eB/\lambda^2 = -6e\lambda d\mu/d\lambda \tag{16}$$

where N_c^∞ corresponds to $\lambda = \infty$, or to Cauchy's constant A. From equation (15) it appears that the sensitiveness $(d\lambda/dN_c)_0$ is inversely as the thickness e and directly as the cube of λ, for a given glass. Since B/λ^2 depends on the refraction of the glass, $d\lambda/dN_c$ varies as λ/e and as $1/(B/\lambda^2)$. It is this feature that makes it ultimately unprofitable to use long columns. No additional sensitiveness is gained; the glass absorbs more and more fully. The columns are not apt to be homogeneous, and the ellipses become excessively small and sluggish in their motion. They offer, however, an excellent corroboration of the sufficiency of equation (1). Writing this in the form (2), equation (17) may be deduced on adding $-1/E\lambda d^2\mu/d\lambda^2$

$$-\frac{d\lambda}{dN_e} = \frac{\lambda^5\mu}{2B}\left\{\frac{\cos R}{e(2B\tan^2 R + 3\mu\lambda^2)} + \frac{1}{3E\mu\lambda^2}\right\} \tag{17}$$

so that the predominating term for long columns is $\lambda^3/6EB$; or for normal incidence, accurately $\lambda^3/6(E+e)B$.

If in equation (13) we convert $d\lambda$ into $d\theta$

$$-\frac{d\lambda}{d\theta}\frac{d\theta}{dN}=\frac{d\theta}{dN}D\cos\theta$$

whence

$$\frac{d\theta}{dN}=\frac{\lambda e}{D\cos\theta\cos R}\left(\frac{\tan^2 R}{\mu}\left(\frac{d\mu}{d\lambda}\right)^2+\frac{d^2\mu}{d\lambda^2}\right) \tag{18}$$

we may compare this deviation with the corresponding case when λ and μ do not vary, but R and N do; *i.e.*, the case of motion along the direction of any given spectrum line. This is a comparison of the vertical and horizontal axes of the ellipses.

42. Equations. Sensitiveness in terms of order.—The quantity $d\lambda/dn$, where n is the order of the fringe of color λ, has subsidiary interest.

Since

$$-\frac{d\lambda}{dn}=\frac{1}{2}\,\frac{\lambda^2}{e\,(\cos R-\lambda\sec R\cdot d\mu/d\lambda)-N}$$

generally, and the condition of centers of ellipses is $d\lambda/dn=\infty$, the coördinate of a center is

$$N_c=e\left(\mu\cos R-\frac{\lambda}{\cos R}\,\frac{d\mu}{d\lambda}\right)$$

If we combine this with the general equation

$$N=e\mu\cos R-n\lambda/2$$

using n_c as the order corresponding to centers of the wave-length λ, and apart from signs,

$$n_c=\frac{2e}{\cos R}\,\frac{d\mu}{d\lambda}=\frac{4eB}{\cos R\lambda^3}$$

In case of the above blue column, for example,

$$e=25.4\text{ cm.};\ B=4.6\times10^{-11};\ R=0;\ \lambda=5\times10^{-5};$$

$$n_c=\frac{4\times25.4\times4.6\times10^{-11}}{125\times10^{-15}}=37{,}400$$

half wave-lengths of path difference in the glass, not including the additional number in the grating.

It is interesting to return to the original equation

$$\frac{d\lambda}{dn}=-\frac{\lambda^2}{2}\,\frac{1}{N_c-N}$$

where $N_c=e\,(\cos R-\lambda\sec R\,d\mu/d\lambda)$ and $-d\lambda=D\cos\theta\,d\theta$, since $\sin i-\sin\theta=\lambda/D$.

If for simplicity we put $I=R=0$ (the actual case with the columns in question, incidence being normal), $\cos\theta=\sqrt{1-\lambda^2/D^2}$ and

$$\frac{d\theta}{dn}=\frac{\lambda^2}{2D\cos\theta}\,\frac{1}{N_c-N}=\frac{\lambda^2}{2(N_c-N)\sqrt{D^2-\lambda^2}}$$

If the center of ellipses is at the E line

$$N_c=N_1 \text{ and } \frac{d\theta}{dn}=\infty$$

To find the size of the fringes at any other line, the D line, for instance, we may again take the example of a blue column (table 10 below) where $N_c-N=0.255$ and put $\lambda=5.3\times10^{-5}$, the grating space $D=2.5\times10^{-4}$, whence

$$\left(\frac{d\theta}{dn}\right)_D=\frac{28.1\times10^{-10}}{2\times.255\times10^{-4}\sqrt{6.25-.28}}=10^{-6}\times22.6 \text{ radians or } 4.6''$$

At the D line, therefore, the distance between consecutive fringes would be less than 5 seconds of arc, showing the diminutive fringes to be expected. After leaving the center the fringes become more and more nearly equidistant. We may therefore estimate, if the angle between the D and E line is here $\theta=4{,}380''$, that somewhat less than $4{,}360/4.6=950$ fringes would be encountered between D and E. They would therefore not be useful, except near the center, where $d\lambda/dn=\infty$.

43. Observations. Green glass column.—In spite of the clearness of the column, the light absorbed at the ends of the spectrum makes it nearly impossible to recognize the small, sluggishly moving ellipses. The observations, therefore, are reasonably good only between the D and E lines. In some cases, moreover, it is easy to mistake the lines, from the coincidence of the direct

TABLE 9.—Green column. $E=22.87$ cm.; $e=0.68$ cm.; $B=4.6\times10^{-11}$; $I=15°$; $R=9.7°$: $\mu=A+B/\lambda^2=\sin I/\sin R$; $(3E+e\cos R+2e/\cos R)=70.66$; $\mu=1.53$.

Lines.	*C–D.*	*D–E.*	*E–b.*	*b–?*	*b–F.*
δN_c	0.185	0.235	0.046	0.112	
	.166	.233	50	.122	....
	.168	.236	48	.065	....
	.175	.237	40	67	....
	.181	.232	.045	.075	....
	.181	....	48	.084	....
	.175	....	43	70	....
	.175	....	45	67	....
	.174	....	43	....	....
	.174	....	....	....	....
Mean δN_c observed	.1754	.2346	.0453	....	....
δN_c computed	.1811	.2347	.0442	....	0.1610

spectra of higher orders, with the two interfering spectra. For work of this kind it would have been preferable to make I (angle of incidence) about 45°. The individual observations are given in table 9. Between D and E the error is about ± 0.002 on a length of $12.6 = \Delta N_c$ for the total displacement; *i.e.*, about 1.5×10^{-4} of $\mu - 1$. In computing δN_c from equation (9), the value of B computed for light crown glass from Kohlrausch's tables was accepted, as a special measurement for so many plates of glass seemed out of the question. The films of Canada balsam are negligible.

The quantity ΔN_c was roughly measured. To find this accurately it would have been necessary to make special adjustments, as it is large ($\Delta N_c =$ 12.6 cm., quite above the range of the micrometer), and as a readjustment of the mirror must be made in the presence and absence of the column, for which it is difficult to make an allowance. The end faces were not quite plane parallel. Using equation (6) the value of the correction $2EB/\lambda^2$ is 0.605 cm., whence

$$\mu = 1 + \Delta N_c/E - 2B/\lambda^2 = 1.525$$

at the D line. The value found directly from the total reflection for a similar glass was 1.521. As without the correction $\mu = 1.551$, the corroboration of the equation is adequate. One may note that $-2B/\lambda^2 = -0.0265$, as a correction of μ, is independent of E, the length of the column. But, for purposes like the present, a small thickness of glass (E about 1 cm. and within the range of the micrometer screw) is preferable, even if the accuracy could be enhanced by using a stronger telescope.

Table 9 shows that the computed values of δN_c happen to coincide with the observed values between the D and E lines. Between D and C, D and b, the results are quite within the errors of observation and satisfactory. The F line was obviously not observed, some other line in this dark part of the spectrum being mistaken for it. Thus the line $\lambda = 49.58$ would give $\delta N_c = 0.108$, the line $\lambda = 50.41$ would give 0.064, each coming close to some of the observations. The results as a whole therefore attest the accuracy of the equation used, as the computed lines are clearly better than the observed lines.

44. Observations. Blue glass column.—Although this column was more colored than the other, the observations were apparently not inferior. Table 10 contains the results. It was just possible to reach the F line, visually. As before, these data reduce largely to the shift from D to E. The column was too long to be compassed by the contact lever used, and the length E given is therefore approximate.

To compute μ from $\Delta N_c = 14.0$ cm., observed, the equation is as before

$$\mu = 1 + \Delta N_c/E - 2B/\lambda^2$$

where the last term is 0.0265 for the same B and λ; hence

$$\mu = 1 + 0.5503 - 0.0265 = 1.5248$$

agreeing sufficiently with the experimental result.

The observed and computed values of δN_c show the same agreement within the errors of observation as before. Differences are due to the value of B used, which is naturally not quite the same as in the preceding case and should here have been about 2 per cent. smaller. Between b and F, $10^6\lambda = 50.41$ would give $\delta N_c = 0.0709$, which was probably observed.

TABLE 10.—Blue column. $E=25.4$ cm.; $e=0.68$ cm.; $B=4.6\times10^{-11}$; $I=15°$; $R=9.7°$; $\mu=1.53$.

Lines.	*C–D.*	*D–E.*	*E–b.*	*b–?.*	*b–F.*
δN_c..........................	0.178	0.252	0.050	0.063	0.157
	.218	.254	53	71	.191
	.196	.255	47	69	....
	....	.257	49	....	....
	....	.256	47	....	....
	....	.254	50	....	....
	....	.256	48	....	....
Mean δN_c observed...........	.197	.2549	.0491	....	.174
δN_c computed..........	.2005	.2599	.0490	....	.1782

45 Observations. Shorter column.—The results for this column are given in table 11. It was less than one-third as long as the other columns, but, absorbing less light, all the lines were seen. The ellipses being more mobile, sharper adjustment is implied; but the F line could not be recognized with certainty and there was difficulty at the C line.

In this case, $\Delta N_c = 3.961$ cm. lay within the compass of the micrometer. The only error therefore is the intermediate readjustment of mirror in presence and in absence of the column. The index of refraction of the D line is thus

$$\mu = 1 + \frac{3.961}{7.1675} - \frac{0.0265}{\lambda^2} = 1.5261$$

which is of the same order as before.

TABLE 11.—Short column. $E=7.1675$ cm.; $e=0.68$ cm.; $B=4.6\times10^{-11}$; $I=15°$; $R=9.7°$; $\mu=1.53$.

Lines.	*C–D.*	*D–E.*	*E–b.*	*b–?.*	*b–F.*
δN_c......................	0.0588	0.0795	0.0147	0.0432	0.0560
	589	790	158	353	558
	591	792	153	.0305	560
	576	785	155	319	571
	.....	784	150	323	548
	.....	.....	.....	.0224	.....
	.....	.....	.....	230	.....
	.....	.....	.....	213	.....
	.....	.....	.....	331	.....
Mean δN_c observed......	.0586	.0789	.0153	....	.0559
δN_c computed.....	.06035	.07821	.01473	....	.05363

The results for this series are scarcely as good as the preceding, relatively, since finer micrometric measurement was required; but, absolutely considered, they are better. Thus between D and E the agreement is within 7×10^{-4} cm. It is obvious that between b and F different lines were sighted, some of them possibly due to superimposed direct spectra. Thus for

$$10^6\lambda=50.41;=49.58 \qquad \delta N_e=0.0213;=0.0359;\ \text{etc.}$$

Both these and other lines seem to have been used.

46. Summary.—The results contained in tables 9, 10, 11, reproduced in fig. 43, show that equation (1) above, or any of its derivatives (4) and (8), gives an accurate account of the motion of the center of ellipses throughout the spectrum, even in case of such extreme conditions as are introduced by glass columns 10 inches or more long. The constants of Cauchy's or any similar dispersion equation may therefore be obtained directly from observations of this character. In such a case a linear interferometer, *i.e.*, one in which $I=R=0$ approximately, would be specially convenient. If δ refers to the difference of the variables for 2 lines λ and λ'

$$\delta N_e=(E+e)\delta\left(\mu-\lambda\frac{d\mu}{d\lambda}\right)$$

which in case of the simple dispersion equation gives $3B(E+e)\delta\frac{1}{\lambda^2}$. As this linear interferometer will have other interesting properties, it has been thought worth while to construct it in connection with the present work.

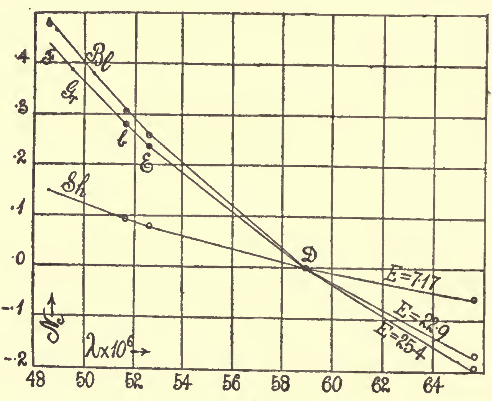


FIG. 43.

The expectation of reaching great sensitiveness by using long columns was not fulfilled in view of equation (17), which shows that the ellipses become more and more sluggish in their motion through the spectrum, as the column is longer.

CHAPTER IV.

PART I.—EXPERIMENTS BEARING ON THE PROPERTIES OF CORONAS.

47. Introductory.—There are a number of obscure points in the theory of coronas when the particles producing them range in size from about 10^{-3} cm. to 10^{-4} cm. in diameter. These relate chiefly to the colored central disks and to the color which for very fine particles spreads uniformly over the white source of light. In the latter case the colors are strictly axial and they suggest the interferences due to thin plates. At least a tentative explanation along these lines seems available.* Light, moreover, is abundantly reflected by the particles, as may be tested by using a Nicols prism. It seems reasonable, therefore, to assume that in spite of their small size the light is also transmitted and that the effect is appreciable when the column of fog is long enough in the direction of the impinging light. All of this is in accordance with the conditions under which axial colors are produced. If they were regarded as diffractions within the geometric shadows of the droplets whose diameter d is decidedly smaller than 10^{-3} cm., the axial distance b in front of the droplet corresponding to the color λ would be $b=d^2/n\lambda$ nearly, for the fringe of the nth order. Hence, even in case of $n=1$, b would be much less than 0.2 mm., whereas the axial colors are seen for all values of b; *i.e.*, they do not vary with b, however large it may be taken.

The disk colors, however, belong to the phenomenon itself. If the elementary equation for a single particle were true, *i.e.*, if

$$\sin\theta=s/2R=C\lambda/d$$

where θ is the angle of diffraction for the wave-length λ and the diameter of particle d, C the constant given by Airy's series, and s/R the aperture of the corona shown by the goniometer, the disks should invariably be white and red edged, as is the case of relatively large particles and small coronas. Actually, however, the white disk is more and more evanescent as d is smaller, the color being particularly vivid in case of the green coronas, where the disk is almost quite green. The disk and annuli thus recall the appearance of the rotary polarization of a quartz crystal cut normal to the axis, though of course all polarization is strictly absent in the colored diffraction phenomenon. I have in fact endeavored to identify the colors by the aid of a rotary polariscope, fig. 44, B and A being the polarizer and analyzer, Q' the quartz rouge, FC the fog-chamber at a distance from Q', Q'' a quartz column sufficiently long to give a white field. Hence the coronas could be seen directly through Q'',

* Barus, Am. Journal of Sci., xxv, 1908, pp. 224–226.

whereas the color from Q' appeared contiguously on the left, the apparatus being used like a half-shade. But the attempt was not of practical value for incidental reasons.

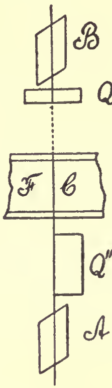


FIG. 44.

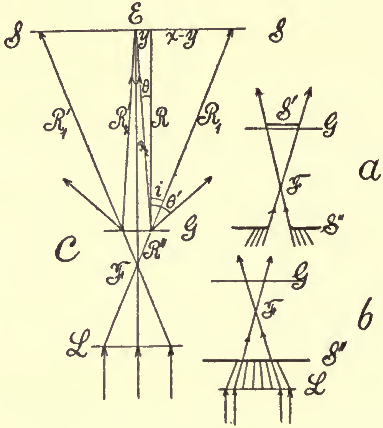


FIG. 45.

48. Experiments with a grating.—Since in the observation of coronas the diffractions toward the right of a group of particles on the left are coördinated with the diffractions toward the left of a group of particles on the right, it seemed interesting to endeavor to reproduce the coronal phenomenon by two identical and coplanar gratings with their rulings in parallel and at a sufficient distance apart. The very large dispersion of the usual commercial grating would here be an annoyance; but the circular grating constructed by Mr. Ives, with 5,000 lines to the inch, is in every way peculiarly adapted for comparison. If such a grating is placed in a cone of somewhat divergent white light from a lens, an interesting succession of colors appears when the axis of the cone intersects a white screen one or more meters off. When the grating is near the vertex, the axis is white; as the distance between grating and vertex gradually increases, the corona shrinks and eventually a colored central disk appears. The reds are vague; but the green, blue, and violet disks, seen in succession, are very strong, each corresponding to a particular distance of the screen. The close agreement of this occurrence with the disk colors of coronas is striking and seemed worth further investigation. It is necessary that the whole grating be illuminated, as the edges are largely responsible for the phenomenon, while the successive annular regions modify it. Hence, an annular method of illumination suggests itself, the annulus being concentric with the circular rulings. In case of sunlight, a breadth of annulus of about 1 mm. for the diameter of several centimeters gave good results.

The adjustment adopted is shown in diagram in fig. 45*c*, where L is the lens about 6 inches in diameter, with its focus at F, about half a meter or more from the lens. G is the circular grating, s the distant screen, several meters off. On the plane G the diameter of the concentric annulus is $2y$; on the plane S the diameter is $2x$, where R'_1 is the distance apart of the planes measured along the white rays, which are elements of a hollow cone with its apex at F. If E is the center of the ring $2x$ and R the normal to G, the rays R'_2 are diffracted to E, and with R'_1 determine the angles θ and i. We may then write in succession, for first order of spectra, if the grating space is D,

(1) $$\sin i + \sin \theta = D/\lambda$$

(2) $$\sin i = (x-y)/R'_1 \quad \sin \theta = y/R'_2$$

If the dispersion y is small compared with the distance R and the incidence i of small obliquity,

(3) $$R'_1 = R_2$$

whence

(4) $$x/R' = \lambda/D$$

Thus the result is independent of the diameter of the annulus $2y$, if R'' is small, and the grating may therefore be placed at the focus F.

The equation may be more correctly written

(5) $$\lambda = \frac{Dx}{R'+R''}(1+y/x)$$

y being always small compared with x. If R is the normal distance between screen and grating, the full expression would be inconveniently

$$\lambda = \frac{Dx}{R}\left(\frac{1-y/x}{\sqrt{1+((x-y)/R)^2}} + \frac{y/x}{\sqrt{1+(y/R)^2}}\right)$$

or if $y=0$, with G at F,

$$\lambda = \frac{Dx}{R}\,\frac{1}{\sqrt{1+x^2/R^2}}$$

In table 12 a series of such measurements is recorded, in which λ is computed from D, x, y, R. Only the outer lines of the circular grating G, fig. 45*a*, were used, the center being rendered opaque by a concentric disk of cardboard S'. The grating was then moved into the divergent cone of white light limited by the circular hole in the screen S'', until the desired color appeared on the distant white screen at the center of the white ring. The diameter $2x$ of the latter is then the only variable. Later an annulus cut in an opaque screen S'', fig. 45*b*, was also used. In such a case, the position S'' determines x, while the position of the grating is of no consequence. The wave-lengths so obtained are as a rule larger than the normal values expected, a result due in part to errors in the judgment of color and in part to the approximate equation used. It is not worth while to enter further into the reason for this, the chief point being that with the use of an annular source all the colors may be made to appear strongly in the center of the figure; *i.e.*, in the center of a white non-diffracted ring. If y is practically zero, these colors will at first sight seem to be axial colors and would appear to be equivalent to those of the coronas of the fog-chamber. This does not, however, seem to be the case, for reasons presently to be given.

TABLE 12.—Disk colors of an annular grating. $D=5\times10^{-4}$ cm.; $R'+R''=700$ cm.; $\lambda=0.714x$ $(1y/x)$.+

Disk, $2y$	Color.	White ring x.	Computed $\lambda\times10^6$.
		cm.	
6.5 cm...	Red	95	70
	Orange...	86	64
	Yellow ...	81	60
	Green....	73	54
	Blue	67	50
	Violet....	57	43
4.5.......	Red	95	69
	Orange...	87	64
	Yellow ...	83	61
	Green....	73	54
	Blue	66	49
	Violet....	59	44
Parallel rays. $x=y$.			
$2y$.	Color.	R.	$\lambda\times10^6$.
4.5.......	Violet....	30	38
6.5.......	Violet....	39	42

It follows from equations (5) and (6) that, since D and R are given, λ varies as x, the radius of the white ring; therefore also with y, the corresponding radius of the white ring at the grating, if the position of the grating is fixed. For $x/R=y/R''$, nearly. Hence, if the outer rings contribute a red center, the inner rings would contribute a blue center, the superposition of all colors being white light; *i.e.*, the usual white disk of these grating coronas. On the other hand, if the outer rings contribute a corona, the inner rings can only contribute blues and violets, for there are no other first-order colors. Hence, greenish-blue and violet centers may be produced from the grating as a whole, without an annular source of light (as is in marked degree the case), whereas the red axial colors occur only when the source is annular.

It is interesting to note that when the central color is violet the reds have already overstepped the center and are approaching the white ring. Fig. 46 shows that when the divergence of the undiffracted white annulus is not too great, it is easy to produce the internal and the external annular spectrum on the outside of the white annulus. In such a case, x' denoting the distance of the red annulus from the center of the screen S, at a distance R from the grating G,

$$\frac{x-y}{\sqrt{1+(x-y)^2/R^2}}+\frac{x'+y}{\sqrt{1+(x'+y)^2/R^2}}=\frac{\lambda R}{D}$$

which for $x'=y=0$, reduces to the above values. If $x=x'$ and $y=0$, *i.e.*, if a colored ring coincides with the white ring,

$$\lambda=\frac{2Dx}{R\sqrt{1+x^2/R^2}}$$

When the red center is in adjustment, the distribution of colors along the axis from screen to grating is naturally in the order of violet, blue, green, etc., and they are relatively dull second-order effects.

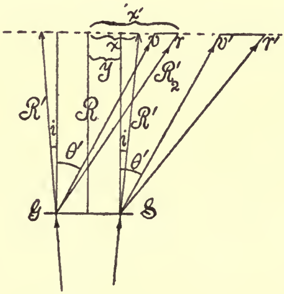


FIG. 46.

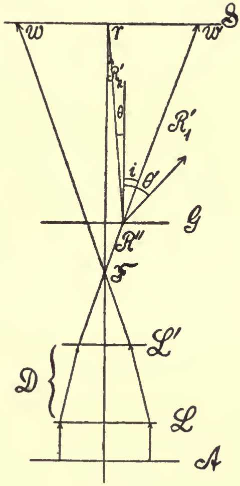


FIG. 47.

49. Experiments with small coronas.—To reproduce these phenomena objectively with coronas is not difficult, so long as the coronas are of small aperture, like those obtained from lycopodium spores. If the adjustment in fig. 45*b* is used, the annulus should be sharp on the screen. The dusted plate G may be placed anywhere near the focus F, showing, moreover, that the thickness of the fog-chamber would not be effective. The best results were obtained by using a pair of lenses of long focal distance, together, as in fig. 47, where A is the annulus, L and L' lenses of focal distances 225 and 120 cm., respectively, G the lycopodium plate, S the white screen at a distance of about 2 meters. By moving L and L' together or the reverse, *i.e.*, by varying their distance D apart, the diameter ww' of the white ring may be varied, while it is kept in sharp focus. For the focal power is $1/F=C-C'D$, C and C' being constant. The white annulus should be about 1 mm. broad and 5 cm. in diameter. Sunlight is preferable. If the ring is too thin, the colors are vague from an insufficiency of light.

If the elementary equation of §47 is used, the results will not differ much from those of the grating, provided the same approximations are made. For if C is Airy's constant, we have in succession, d being the diameter of the particle,

(1) $$\sin(i+\theta)=C\lambda/d$$

(2) $$\sin i=(x-y)/R', \quad \sin\theta=y/R_2$$

(3) $$\lambda=\frac{d}{C}\left\{\frac{x-y}{R'_1}\sqrt{1-y^2/R'^2_2}+\frac{y}{R'_2}\sqrt{1-\frac{(x-y)^2}{R'^2_1}}\right\}$$

If y/R and $(x-y)/R'$ are small in comparison with 1 and $R'_1=R'_2$, practically,

(4) $$\lambda=\frac{dx}{CR'}=\frac{dx}{C(R'+R'')\left(1-\frac{y}{x}\right)}$$

or if $y=0$ and the plate G is at F,

$$\lambda=dx/CR'$$

Using the equation with lycopodium, a difficulty experienced is because the colors, on projection, are reddish-brown or vague. But on reducing the reddish center at r to a dot, and treating it as a blue minimum, $\lambda=49\times10^{-6}$, the following values for the diameter of the spores were obtained:

C	R'	x	d
1.22	250 cm.	5.7 cm.	0.0026 cm.
		5.5	.0027

This seemed to agree with the size of the particles seen under the microscope. The diameter usually quoted is 0.0032 cm. The question would resolve itself into a decision of the color, which is here a minimum and is therefore of no further interest. If we regard the red edge of the disk as a blue minimum and use the goniometer of chord s and radius $R=30$ cm., $s/2R=1.22\lambda/d$, whence $sd=73.2\lambda=0.0036$. In my experimental work with fog particles I used a somewhat smaller constant, $ds=0.0032$.

50. Experiments with large coronas, annular source. Coronas by reflection.—The coronas due to water particles are not intense enough to be available for projection, unless perhaps a fog-bank more than 12 inches thick is used. The screen has to be placed so close to the fog-chamber (R very small) and the color varies so rapidly along R, that the semblance of axial color (angle of diffraction zero) found above, quite vanishes. The particles as a rule range in size from somewhat above 10^{-3} cm. to somewhat above 10^{-4} cm., after which the coronas become filmy. The work must therefore be done subjectively, as usual, and the direct light to the eye may be blotted out by a small screen or disk, placed between the point source and the eye. In such a case the colors beyond the edge of the disk are very vivid.

When an annular source is used, the path of the cone or cylinder of light through the fog is highly illuminated and the diffraction is both inward and outward from this shell. The shadow of the disk of the annulus, therefore, necessarily remains permanently dark, so that this should be small and the cone as divergent as the fog-chamber permits. The colors are quite brilliant, in spite of the fact that fewer particles are used than in the case of an unobstructed point source.

In fig. 48, if *FF* is a fog-chamber, *ww* and *w′w′* elements of the conical shell of light, a position of the eye of *E* behind the fog-chamber may be found, at which the whole interior of the cone flashes into the uniform color of the disk. For this purpose i must be small or zero, or the shell nearly cylindric, in which case, if R is the distance of the eye from the center of the chamber, s/R the aperture of a given color minimum, the elementary equation becomes

$$ds = 2.44\lambda R\sqrt{1-(s/2R)^2}$$

In this case s is constant and R is variable. It is found difficult to use this method practically.

An interesting accompaniment of these experiments is the occurrence of vivid color when the eye is in a proper position on the illuminated side of the fog-chamber, as, for instance, at E' or E''. In other words, there is also vivid diffraction in connection with the reflected light, a phenomenon which it is difficult to detect in case of the absence of the annular screen A, since all the fog is illuminated. These colors come from both the outside and the inside of the cone. When the colors fade, or when there is much directly reflected light, they may be restored by a Nicols prism, as this on orientation cuts off the reflected light only. The diffracted colors are in no case polarized, whether seen by transmitted or reflected light.

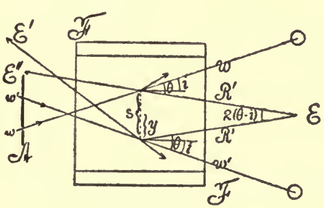


FIG. 48.

Experiments made at some length with these annular sources, supplied with both polarized and unpolarized light, did not lead to further results worthy of note. They showed clearly that the axial and disk colors can not be explained as suggested and produced above, in case of the grating, however many points of resemblance there are. In fact, if with regard to fig. 48 we write

$$\sin(i+\theta) = C\lambda/d$$

which gives a rough reproduction of the facts, the aperture of a corona of average wave-length when the particles approximate 10^{-4} cm. must soon exceed 90°, so that the colors are spread out along the axis of a shallow cone close to or within the fog-chamber. It would require a similarly divergent cone of light to make these colors seem axial. Now, in producing coronas from a single small source of light, the rays which strike the chamber are quite inadequately divergent.

Finally, by putting the electric lamp with an annular screen close to the fog-chamber, thus avoiding all lenses, the phenomena were produced much more brilliantly, but without new results.

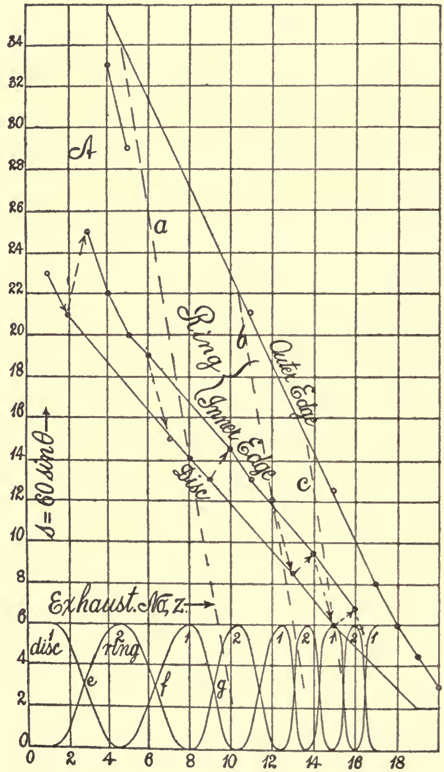


FIG. 49.

51. Coronas from a point source.—The best evidence obtained bearing on the nature of the disk colors is probably found in my own earlier experiments with strictly homogeneous light. The green light of a mercury lamp is used for illumination, as the source must be very intense. The results obtained are surprising and are summarized in fig. 49 in which the ordinates s are the chords of the radius of $R=30$ cm. subtending the aperture 2θ of the coronal disk or ring specified, so that $s/2R=\sin\theta$. The abscissas are the ordinal numbers, z, of the successive partial exhaustions, all of them identical. Arranged in this way, all the apertures s, curiously enough, vary linearly with the number of the exhaustions z, while the fog-particles are gradually increasing in diameter d, exponentially. If the quantity s were laid off in terms of d, the curves would be roughly hyperbolic, and less serviceable for exposition. The three curves selected from many similar results refer, respectively, to the aperture of the edge of the green disk, and to that of the inner and outer edge of the first green ring, the latter value of s being about twice as large as the s for the edge of the disk. During the earlier exhaustions, $z=1$, 2, 3, etc., the coronas are very large and filmy, and a sharp value of s is out of the question. Consequently observations in the regions A can not be expected to agree closely. The graph for the outer ring, moreover, had to be taken from a separate series of observations.

The feature of these experiments is this, that the disk and the first ring are alternately vividly colored (green) and alternately dark (yellowish, due to the dull mercury line); *i.e.*, when there is interference in the disk there is reinforcement in the ring, and *vice versa*. When the number of the exhaustion z is high and the fog-particles therefore large, these alternations are crowded so closely together that the successive partial exhaustions necessarily skip one or more cases; but for the smaller particles, *i.e.*, as far as exhaustion $z=15$, the results are invariably definite. The curves for the disk and inner edge are joined by arrows, showing the successive position of vivid green color.

There are two points of view from which these results may be interpreted. We may lay off the intensity of green color in the disk and ring separately, in which case maxima of the one will coincide with minima of the other as shown at the bottom of the chart, fig. 49, 1 referring to the disk, 2 to the ring. Again, the phenomenon may be regarded as a series of contracting rings, which are first seen in a ring and successively in the disk. Lines a, b, and c have been drawn from this point of view. The latter can not, however, be correct, since it frequently happens that vivid green color is absent *both* from the ring and from the disk, a condition not suggested by the lines a, b, c. On the other hand, the lower intensity curves of the figure call for just this result at e, f, etc., so that the former point of view is in accordance with observation. Apart from the outer rings, which are too large for observation, we may even add that the axial colors treated at the beginning of the paragraph probably again reverse the alternations of the disk, or again approach the case of the ring, though direct observations on this feature were not made.

In fig. 50 the same attempt is made to represent the periodicity in disk and inner ring, by representing the apertures of coronas

$$s=2R\sin\theta;\quad R=30 \text{ cm.}$$

in terms of the diameter d of the cloud particles, computed from the successive exhaustions. Here one may average the diameter corresponding to vivid green color in the order

Disk, $10^4 d =$	1.2	2.5	4.0	5.5	7.0?	8.5? cm.
Ring, $10^4 d =$	1.8	3.2	4.7	6.5	7.7? cm.	

where above $10^4 d = 6.5$ cm., the removal by exhaustion is initially too rapid to catch the successive cases of maxima. If we also remember that in the first exhaustion the coronas are filmy, it is thus a plausible result of observation that the vivid green disk reappears whenever the diameter of particles increases by a quantity less than 1.5×10^{-4} cm. and greater than 1.2×10^{-4} cm.

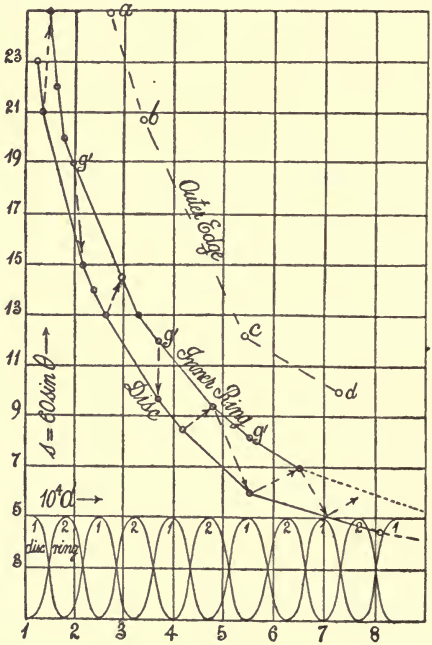


FIG. 50.

If the prolonged occurrence of vivid color throughout several exhaustions at first, and the unavoidable accumulation of errors in the successive exhaustions toward the end of the series, be taken into consideration, it is not unreasonable to adjust the maxima of disk color for diameters of fog-particles increasing in multiples of 1.3×10^{-4} cm. as follows:

$10^4 d =$	1.3	2.6	3.9	5.2	6.5	7.8 cm.
No.	1	2	3	4	5	6

It is certain, moreover, that the first of these coronas is essentially the first in order, inasmuch as no other corona can precede it. It would follow that the reddish coronas first visible when white light is used would correspond to particles ($\lambda = 63 \times 10^{-6}$),

$$d = 10^{-4}\frac{54.6}{63}1.3 \doteq 1.12 \times 10^{-4} \text{ cm.}$$

or not below 10^{-4} cm. in diameter.

In a general way these conclusions agree with the datum estimated for the axial colors of coronas, where axial yellow, corresponding to second green coronas, was referred to particles of the order of size $10^4 d = 2.2$ cm. In both cases the assumption is made, of course, that all nuclei are caught in the exhaustion. In fig. 51 I have shown an attempt to regard the middle of the first ring as the minimum corresponding to the disk, as the intensities alternate. In the graph the abscissas denote the diameter of fog-particle in 10^{-4} cm., computed from the successive exhaustions. The ordinates of the curves *a*

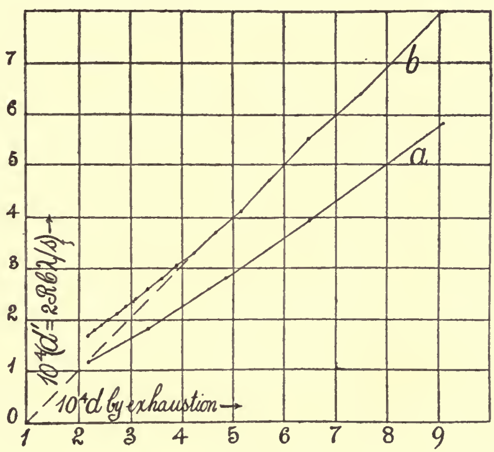


FIG. 51.

then indicate the diameter of particle if the aperture *s* of the middle of the first ring is taken in the equation $d = 2RC\lambda/s$; whereas the ordinates of *b* show the diameter of particle, when the aperture is measured as far as the middle of the dark zone between the disk and first ring; *i.e.*, midway between the edge of the disk and inner edge of the first ring. Clearly the curve *b* is preferable. As far as particles of the order of size $d = 3 \times 10^{-4}$ both methods (curve *b*) agree within the errors of observation; but for particles less in diameter, the data of the optical method are increasingly too large as the particles become smaller.

If partially monochromatic light, like that of the arc lamp filtering through ruby glass, is used, the graph obtained on successive exhaustions is sinuous, indicating the alternate illumination of the disk and first ring; but it is not sharp and clear, as in the case of strictly monochromatic light.

PART II.—DISPLACEMENT INTERFEROMETRY WITH FILM GRATING.

52. Introductory.—The transparent plate grating is expensive and relatively unavailable, owing to the fact that the glass soon injures the diamond edge of the ruling machine. It is therefore desirable to attempt to replace it by the film grating, now so admirably made by Mr. Ives and others. Some time ago I showed that this is quite possible, though the ellipses obtained were not comparable in definition with those of the ruled glass plate. The following paragraph is an attempt to improve the former method.

The arms of the interferometer used were nearly 150 cm. long. Hence the tests made are throughout severe. It is shown in the following paragraphs that though any kind of film grating may be used, the particular type in which the smooth side of the film is cemented to plate glass while the ruled side is exposed is nearly as serviceable as the ruled plate grating.

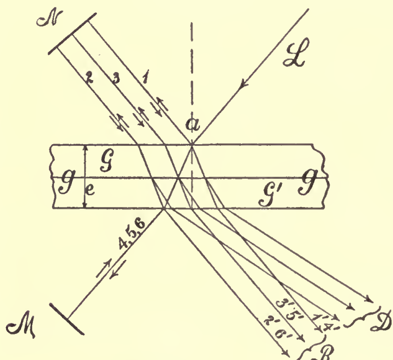


FIG. 52.

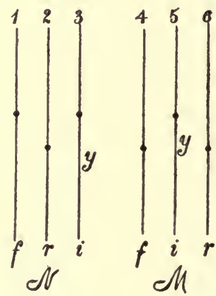


FIG. 53.

53. Films between glass plates.—This is the usual form of the grating in the market, the unruled side of the film (15,000 lines to the inch) having been attached with Canada balsam to one of the plates, while a thin film of air separates the ruled side from the other plate. If this grating is used in the interferometer there must be three reflections on one side of the grating and three on the other, supposing that the ruled face of the film and the glass face are practically continuous.

Fig. 52 represents the case when the rays return upon themselves. The white rays L from the collimator are separated into the groups 1, 2, 3, reflected respectively from the front, the rear, and the intermediate faces of the grating GG' (gg being the film), toward the opaque mirror N, whence they return into

1′, 2′, 3′, to enter the telescope. Similarly the transmitted rays 4, 5, 6, after reflection from the opaque mirror M, are separated into the three groups 4′, 5′, 6′ reflected from the front, the intermediate, and the rear faces of the grating towards the telescope. It follows, therefore, that the ray 6′, reflected from the rear face only, can have no spectrum, since it does not pass through the film g. The others, 1, 2, 3, 4, 5, do pass through the film, and when superposed in pairs must give rise to elliptic interference in the superposition of the spectra. The figure shows the refracted rays R and the diffracted rays D, just before entering the telescope.

Since the glass is ordinary plate, it is almost invariably slightly wedge-shaped and consequently direct rays will be seen in the telescope as six parallel lines. These are shown in fig. 53 in the form actually observed with the present grating. The rays 1, 4 from the front, 2, 6 from the rear of the plates, are dazzlingly white, whereas 3 and 5 from the film within are respectively brownish-yellow and yellowish. The latter are thus easily distinguished, but are otherwise adequately white and perfectly sharp. It follows also that the air-film is plane parallel, otherwise there would be two images from this region; that it does exist, however, is shown by the reticulated interferences below, §§ 55 and 56.

If the group of rays 1, 2, 3, is separated, all but one may be screened off, and this is frequently possible. It is much more difficult, however, to screen off the rays 4′, 5′, 6′ individually as a rule, though if the angle of incidence is large (45° as compared with 15° in this paper) this may also be done. No. 6′ is of no consequence, since it does not appear in the spectrum; No. 4′ may be removed by blacking a vertical line at the point a in the diagram; or a narrow screen (vertical rod) may be placed at the proper position just behind the grating. The object in view is to admit only the two spectra which are to interfere in the telescope, as this sharpens the black lines enormously.

It is for this reason that it is undesirable to have the rays return upon themselves; *i.e.*, they should not be reflected from M and N quite normally. In such a case, as will presently appear more clearly, the blotting out of individual rays or spectra is more easily accomplished.

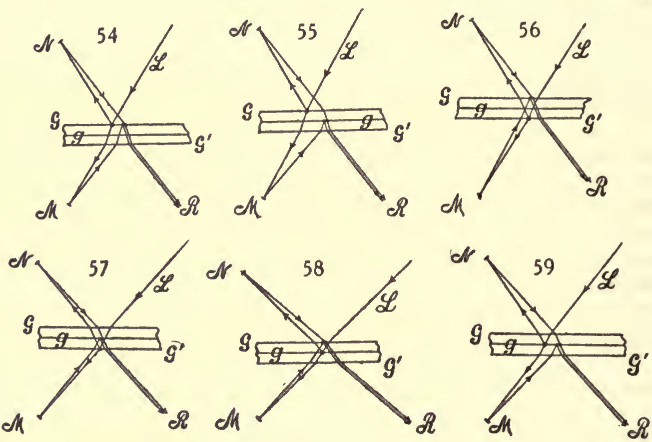
In order that the slit images may be superposed horizontally and vertically, a fine wire is drawn across the slit of the collimator, the wire being imaged by the black spot on each line, as suggested in fig. 53. The adjustment screws (horizontal and vertical axes of rotation) on the mirror then enable the observer to bring any two slit images and the corresponding spots into coincidence. This adjustment must be made accurately, if the interferences are to be seen in the spectra in the field of the other telescope.

54. Continued. The groups 1+4, 1+5.—The character of these interferences, in which the rays 1 and 4 or 1 and 5 are superposed horizontally and vertically, is shown in figs. 54 and 55. The rays do not return upon themselves, L being the impinging white vertical sheet of light, M and N the opaque mirrors, g the film inclosed between the plates GG' which are equally thick.

In the case 1 + 4, the component rays pass through the grating, one and three times, respectively. The corresponding path difference may be called 2δ. The micrometer position of the mirror M was at the arbitrary mark 0.525 cm. The interferences were very fine hair-lines, evidencing a very distant center. They had the usual tendency of revolving from vertical to horizontal when they are thickest, into vertical again, when M moves; *i.e.*, enormous concentric equidistant circles are moving through the field horizontally, the center being invisible. The data correspond roughly to the horizontal at the sodium line.

Case 1+5 showed the same phenomenon of fine revolving hair-lines, seen to be circles when horizontal. Their micrometer position, however, was 1.275 cm., roughly, so that the displacement of M of 0.75 cm. has intervened. The path difference is δ.

Neither of these cases is of value unless the centers happen to be close at hand. This is a matter of chance residing in the wedge shape of the plate.



FIGS. 54 TO 59.

55. Continued. The groups 2+4, 2+5.—Both of these interference patterns are interesting, as the lines are strong or centered ellipses. The nature of the interferences is given in figs. 56 and 57 with the same notation.

In case 2+4, both component rays pass through the glass twice. The path difference is therefore $\delta = 0$, and the micrometer position of M, 2.025 cm., an advance of 0.75, as before. The system is self-compensated. The fine, very strong, revolving lines, circular when horizontal, are *reticulated* (the center, however, not being in the field), showing a second set of interferences not nearly so strong, crossing the former set usually at about 45°. The fainter lines may be obtained by slightly revolving the mirror M about the vertical axis by aid of the adjustment screws. These reticulated forms occur only for the compensated adjustment. They are due to the two reflections possible on the two sides of the very thin air film. Probably the reflections from the film to glass and return constitute the weak lines, the strong lines not being reflected at the film, but directly diffracted and transmitted. The reticulation does not interfere with the sharpness of the strong lines.

The case 2+5 appears for the micrometer reading at the mirror M, 2.775 cm., a final advance of 0.75 cm. Since the group 2 passes through the glass twice and the group 5 but once, the path difference is now $-\delta$. The path difference -2δ will not occur, as the ray 6 has no spectrum.

The present superposition 2+5 gives the only ellipses obtainable with their centers in the field. They are sharp in line, but usually somewhat weaker than desirable in practice, unless the other spectra are blotted out as suggested in § 53. In such a case they become quite as available as the ellipses from a plate-glass grating.

56. Continued. The groups 3+4 and 3+5.—Both of these (see figs. 58 and 59) are interesting in case of the given grating.

The case 3+4 appears when the micrometer reading at M is 1.275 cm., seeing that whereas one component ray passes through the glass 3/2 times, the other passes but 1/2 times, the rays starting in the middle. The path difference is thus again δ, but the interferences are thoroughly different, naturally, since the front glass plate is in question. They are very coarse, strong, revolving lines, curved when horizontal. The center is not far distant, but outside of the field. What is very striking is the rapidity of their revolution. They pass, almost at once, from vertical to horizontal and back to vertical again, and are thus a sensitive criterion for the position of the mirror M on the micrometer.

Finally, the case 3+5, since each of the component rays passes through the glass once, reproduces the compensated position $\delta = 0$, with the micrometer, M at 2.025. The fringes are the same strong reticulated set described above. Both reflections of the component rays take place at the same face.

57. Centers of ellipses.—To bring the centers of the ellipses into the field if the former are near at hand, the observing telescope may be raised or lowered, provided of course the vertical extent of the entering pencil of light is larger than the diameter of the objective.

If the center is far removed, however, the grating and simultaneously the mirror N must be correspondingly inclined, so as to bring the entering pencil of light from N back again into coincidence with the pencil from M. Thus the coarse fringes with M at 1.25 cm., when thus explored, are found to be very eccentric ellipses with the long axis vertical, which accounts for the rapid rotation mentioned above. In other words, when the spot of light on N is near the top (grating and N being reciprocally inclined), the field intersects the top end of the major axes and the curvatures may be seen. When the spot on N is near its bottom, the lower ends of the major axes may be seen. Finally, when the spot on N has an intermediate position, the lines no longer rotate, but pass from fine, through coarse, into fine again, when M moves, indicating that the horizontal minor axes of the ellipses now occupy the middle of the field of the telescope. Scarcely any curvature of the lines can now be seen. In the first position (top of ellipses visible), 0.0001 cm. produces very visible change of inclination of the nearly horizontal lines.

With the very fine lines even this adjustment fails to bring the center of ellipses into the field. Thus, in the compensated case, $2+4$, with M at 2.025 cm., the spot of light on the opaque mirror N may (with simultaneous inclination of the grating) be moved fully 10 cm. vertically, without producing marked effect on the interference pattern. The lower parts of enormous circles are intersected by the field of the telescope throughout.

58. Film or ruling on one side of the glass plate. Ruled grating.—If the film is on one side only, the case of the plate-glass grating is reproduced. From the absence of so many superposed spectra, the results should therefore be better.

The plate-glass grating, 10,000 lines to the inch, mounted in the interferometer with its ruled face toward the collimator, shows but two lines for the reflection from each of the opaque mirrors, of which three only give rise to spectra, the beam of the fourth being reflected from the rear face and not passing through the grating, after reflection. But this case may be reached by reversing the grating (ruled face toward the telescope), under which conditions all the rays have spectra. The micrometer positions for these interferences were as follows:

M at 2.95 cm. Ruled face in front. Reflection from the front face. Path difference δ. Reflection from the rear face may be screened off at the opaque mirror N. Fine, sharp, solitary ellipses, very black lines on an even, brilliantly colored ground. No stationary interferences from the front and rear faces of the grating.

M at 1.90 cm. Ruled face front or rear. Reflection from front and rear faces conjointly. Compensated position. Path difference $\delta = 0$. Fine revolving hair-lines belonging to a remote center.

M at 0.85 cm. Ruled face to the rear. Path difference $-\delta$. Fine ellipses, but complicated by stationary interferences and superposed spectra, which can not be easily screened off. The first position is here also available.

The position (2.95 cm.) with ruled face toward the collimator, is thus the practical case, since it is sharpest and without complications.

The micrometer displacement ΔN, which should correspond to the extreme positions of the grating (ruled face forward and ruled face rearward), if e is its thickness, I and R the angles of incidence and refraction, μ the index of refraction, is

$$\Delta N = 2e\mu \cos R$$

Here roughly $I = 15°$; $R = 9.7°$; $\mu = 1.53$; $e = 0.67$ cm. Hence

$$\Delta N = 2 \times 0.67 \times 1.53 \times 0.986 = 2.02 \text{ cm.}$$

The difference (observed value 2.1 cm.) is due to the necessary reversal of the grating, which requires considerable readjustment. Using the third position, the observed ΔN was more closely 2.059 cm. But here again two separate pairs of slit images must be placed in coincidence, which calls for a double rotation of the mirror M.

59. Continued. Film grating not cemented to glass.—To test this case for the film grating, a number of fine specimens kindly made for me by Mr. Ives were at hand. The film of these was not attached continuously to the glass by a layer of Canada balsam. Nevertheless the film was perfectly smooth and reflected well. It had about 15,000 lines to the inch, which is rather too great a number for work of this kind, as the ellipses have their long major axes horizontal. Accuracy is enhanced if the major axes are vertical. Furthermore, the spectrum toward the right of the refracted ray is too far distant from the micrometer for the easy manipulation of the latter. Hence the spectra toward the left must be used, if available.

The refracted slit images are shown in fig. 60, when the ruled face is toward the rear. There are but two lines visible from the opaque mirror M and three from the opaque mirror N, the fainter, yellow and less even line, No. 3, coming from the film. To obtain the ellipses which are here alone sought, lines 2 and 3 reflected from the same face are put in full coincidence. The micrometer positions were approximately as follows:

M at 2.9 cm. Film toward the collimator. Good flat ellipses, but not very strong. Path difference δ.

M at 2.0 cm. Compensated position. Path difference $\delta = 0$. Fine hair-lines, revolving.

M at 1.1 cm. Film toward the telescope. Path difference $-\delta$. Ellipses resembling the first case.

The stationary interferences were present, but not objectionable. The spectrum, showing a very bright sodium line, is brilliant.

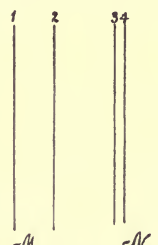


FIG. 60.

A large number of other film gratings of the same kind were tested, but with no further results. Sometimes there are four slit images, at other times six, depending upon the adjustment and shape of the films. The tests were somewhat severe, as the arms of the interferometer were nearly 150 cm. long; but it is clear that in order to get the best results the film should be continuous with the glass, being cemented with Canada balsam on the unruled side, while the ruled side is outermost. In this case the plate grating is almost reproduced.

60. Single plate, film grating.—It has been stated that the best results would be obtained with a film grating cemented to plate glass on the smooth side, with the ruled side exposed. Having failed to make one adequately plane myself, I was fortunate in securing a sample through the courtesy of Mr. Ives. The number of lines to the inch, 14,438, was rather in excess of the number desirable for interferometry, as the ellipses when the number of lines exceeds 10,000 to the inch are liable to be too broad for accurate measurement. Otherwise the grating sufficed the required purposes admirably. The great advantage in adjustment in case of such a grating is the presence of only one strong reflection, namely, that from the uncovered face of the glass. The side covered by the grating reflects but very feebly. Consequently the grating is to be mounted with the rear side (smooth glass) toward the source of light and the celluloid side toward the eye of the observer. Hence if the two strong slit images in the telescope of the interferometer are placed in contact horizontally and vertically, the ellipses are found for approximately equal distances, without difficulty. They are centered, since both reflections occur at the same position of the same face. It is also possible to obtain faint reflections from the grating face, the slit image being usually deep blue in color. In spite of this, however, the spectrum (as the ray does not again pass through glass) is strong. The interferences are rarely centered, as the two reflections contain the angle of the faces of the glass plate between them. They consist of lines coarsening and rotating 180°, as the vertical projection of the distant center is passed. The ellipses, if too broad, may as usual be made smaller with a thick compensator, but at a sacrifice of sensitiveness. With concave mirrors, on the horizontal pendulum for instance, the ellipses are apt to be small and round, even if flat and coarse with plane mirrors. Thus with concave mirrors the interposed thick plate of glass is not needed.

PART III.—ELLIPTIC INTERFEROMETRY WITH A NERNST FILAMENT.

61. Introduction.—The ideal illumination for the present purposes is sunlight, inasmuch as the lines of the spectrum are always present; but it is not generally available. In its absence the electric arc does good service. Here the sodium line is always visible and of sufficient intensity to be utilized as a landmark, if desirable. It is preferable to use the arc without condensers and to place it several feet from the slit, in order that the rays may be nearly parallel and that the pencil which comes out of the objective of the collimator may be a nearly vertical sheet, converging toward the opaque mirrors. Such a rapier-like beam is more easily made to penetrate long tubes and similar appliances. If the interferences are to be sharp, the lateral extent of the pencil must be as narrow as practicable. Moreover, while the rays from the collimator are parallel in their horizontal projection, they are not so in their vertical projection, for which case the focus may as a rule be advantageously placed (by moving the electric arc to or from the slit) at the opaque mirrors of the interferometer.

The arc lamp has one great objection, however, inasmuch as the mobile arc requires constant attention, and even in such a case adequate illumination often fails at a critical moment. It is therefore desirable to find a more steady source of illumination of sufficient intensity, and this is clearly attainable with the use of the Nernst burner. Experiments were therefore made with this light and the following compact form of apparatus which satisfies many purposes and suggests itself particularly with a view to the photography of the interferences, such as may be needed in the next section.

62. The Nernst burner.—In fig. 61 (horizontal projection), N is the usual type of Nernst burner with electromagnetic base, the filament being at a. It is inclosed in the rectangular case A, of blackened tin-plate, judiciously provided with holes for ventilation, so as not to allow the appreciable escape of light from the case, except in front. The front is a micrometer slit S, in advance of which a short end of flanged tubing c is adjustable, so as to cut off the excess of light spreading up and down. The plate holding the slit S may be removed by sliding it up on guides. Similarly the slit S and the tube c may be removed by sliding the latter horizontally on this plate. O is the lens of the collimator, aO being (nearly) its principal focal distance. This is found by obtaining the sharp image of the slit, reflected by the opaque mirror N on the jaws of S.

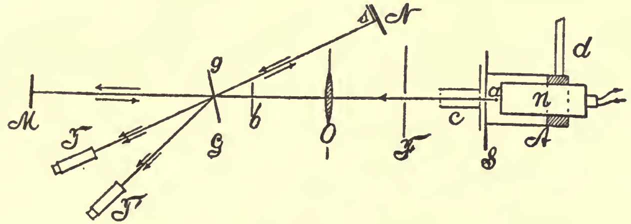


FIG. 61.

In order that the interferences may be sharp, the beam of light falling on the grating must be of slight extent (⅛ inch) laterally. For this purpose a screen with an appropriately wide vertical slit is placed either at F or on the grating table at b. When this is done two slit images may usually be seen at the mirror N and the brownish one is screened off there. To obtain the solitary ellipses, the ruled side g of the grating should face the source of light. As the grating is of ordinary glass plate and therefore wedge-shaped, the top and bottom of the grating should be selected so that the wedge and the thickness effect act in concert, to separate the two slit images at N, referred to. In this case the undesirable image may be more easily screened off.

63. Remarks.—It was my expectation that the Nernst filament might itself be used as a slit without further appliances than the screen tube c. But this is not adequately the case, as the filament is a little too thick. Without the slit and the tube only, the ellipses are just suggested. Possibly if the white porcelain surface of the Nernst burner were black instead of white porcelain, clearness would be enhanced. But the ellipses would not be useful for measurement. Without the slit, but with the slotted screen T or b, the ellipses are strong but somewhat washed, so that the fine lines to right and left soon vanish. The rings could actually be used for measurement, for the centers are well indicated and the motion of rings adequately clear. With the slit and tube, or slit tube and screen, the ellipses become sharp and the fine lines indefinitely visible. The slit need not be very fine, but as it is finer the velvety black lines on the colored spectrum become more marked. The interference pattern is now quite as good as with the arc lamp.

Naturally when the filament is so near the slit, the rays on leaving the collimator diverge strongly in the vertical plane. Hence the illuminated parts of the mirrors M and N may be 2 or 3 inches long. If these mirrors are of ordinary plate glass it is not liable to be adequately perfect over the whole length, and the ellipses will be imperfect in form. But this is not a serious disadvantage.

At wider ranges (g to M and N 1 to 2 meters), the arrangement is not very satisfactory for photography, because the light passing through the telescope, unless the objective is very large (a ¾-inch objective was used), is only a small part of that passing through the slit. Hence the light camera at the end of the telescope is insufficiently illuminated. For photographic purposes it would then seem to be better to place the Nernst filament at a distance from the slit and to use a condenser; but I was unable to obtain marked advantages in this way, while the condenser is an annoyance. Hence for photographic purposes it is better to replace the plane mirrors M and N by identical concave mirrors in which the light is appropriately condensed. This is done in the inclination apparatus in Chapter I, § 21, and further reference has been made there. In any case, however, greater steadiness and freedom from tremor than the laboratory affords would be desirable for photography, and though it is not difficult to obtain families of ellipses in the way given on the ground-glass screen of the camera, few experiments in actual photography were made.

The spectrum of the Nernst filament is free from the Fraunhofer lines. It is, however, easy to obtain the reversed D lines, by using an ordinary sodium flame placed either in front of the slit or (contrary to expectations) even behind it within the collimator. One would have expected the latter method to interfere with the definition, but it does not seem to do so. When the sodium lines have once been indicated, the cross-hairs of the telescope may be placed in coincidence with them and the desirable fiducial lines of the spectrum thus obtained.

PART IV.—SCATTERING IN THE CASE OF REGULAR REFLECTION FROM A TRANSPARENT GRATING, AN ANALOGY TO THE REFLECTION OF X-RAYS FROM CRYSTALS.

64. The phenomenon.—No doubt the following phenomenon has been noticed before, but I have seen no description of it. If a vertical sheet of white light L, from a collimator, is reflected from the two faces of a plate-glass grating, having about 10,000 or more lines to the inch, g being the ruled face, the two beams b and y going to the opaque mirror N are respectively vividly blue and brownish yellow. In other words, more blue light is regularly reflected from the ruled surface than is transmitted, and more reddish light transmitted than is reflected. Since the plate grating is not quite plane parallel, two of the four rays, b' and y', are seen in the same colors in the telescope. This is a great convenience in adjusting the displacement interferometer, where the spectra from b alone are wanted, and the y ray may be screened off at N, while the other y' has no spectrum.

The transmitted rays t after reflection show very little difference, the one reflected at g being perhaps slightly yellowish as compared with the other.

The spectra from b and y, if compared one above the other, are practically identical. The difference is not sufficiently marked to be discerned by the eye. Multiple reflection from the two faces gave no further results.

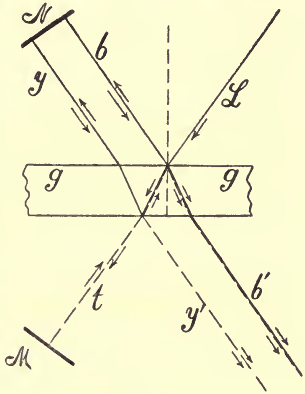


FIG. 62.

Finally, to be colored blue, the beam must be reflected from the air side and not from the glass side, where but little appreciable effect is produced. If the grating is turned 180°, both the b and y rays are nearly white, while the t rays now correspond to the b and y rays and are vividly colored.

Outside the ruled surface and with any ordinary unruled plate of glass, all images are of course white. I mention this merely since one might suppose the absorption or color of the glass to have something to do with the experiment. The film grating, where sharp reflection takes place from the glass and not appreciably from the film, does not ordinarily show the phenomenon; but in case of the single-plate film grating of paragraph 60, it is astonishingly strong in the refracted slit images seen in the telescope. These are, in fact, azure blue when coming from the mirror N and reflected from the front side (toward the lamp) of the film; deep brown when reflected from the rear side, after having passed through the film. The two images may be superposed by rotating M with the production of nearly white light. Moreover, the marginal light (otherwise identical but not passing through the film) is white. The images in question are sharp, but it is possible that the material of the film may somewhat contribute to the color.

65. Explanation.—Scattering is usually and perhaps essentially associated with diffuse reflection. The present phenomenon, however, is strictly regular reflection; *i.e.*, there is a wave-front, for the blue and yellow slit images are absolutely sharp in the telescope. This is the interesting feature of the phenomenon, which associates it at once with the recent famous discovery of Friedrich, Knipping, and Laue relative to the reflection of X-rays from the molecular planes of crystals, and it is for this reason that I call attention to it.

In case of the grating the sources of scattered light-waves are not only identical as to phase, but these sources are at the same time equidistant. Hence collectively they must determine a wave-front of somewhat inferior intensity but otherwise identical with the wave-front of normally reflected or diffracted light; *i.e.*, the wave-fronts of regularly reflected and scattered light are superposed.

Moreover, if the grating is turned in azimuth even as much as 45° on either side of the impinging beam (after which the many reflections and diffractions seriously overlap) the blue and brown colorations are distinctly intensified. This also is in accordance with anticipations, for the number of lines which are comprehended within the lateral extent s of the narrow beam L as the angle of incidence i is varied, increases as $s \sec i$; whereas the lateral extent of the reflected beam is no larger than that of the impinging beam. Hence there should be increased intensity of scattered light in the ratio of $\sec i$ or increasing markedly with i from 1 for $i=0°$, to ∞ for $i=90°$. In other words, the scattering lines of the grating are virtually more densely disseminated when i increases.

For the light reflected from the inside of the glass plate the evidence to be obtained from color in case of the ruled grating is too vague to admit of definite statements. I have not, therefore, attempted it.

66. A further analogy to the reflection of X-rays.—With regard to the recent experiments (*l.c.*) on the reflection of X-rays from crystals, it may further be interesting to recall my experiments (Phil. Mag., XXII, p. 121, 1911) on the interferences produced by two identical but separate halves of a reflecting grating, with the rulings parallel and originally in the same plane. The interferences observed are brought out by moving one of the half gratings micrometrically parallel to itself, to the front or to the rear of the other half, and are here necessarily linear and parallel to the rulings. If i (angle of incidence) $=\theta$ (angle of diffraction) and d is the normal distance apart of the gratings, the same equation $n\lambda = 2d \cos \theta$ holds. In other words, two identical spectra originating in parallel planes, at a distance apart commensurate with the wave-length of light, are superimposed throughout their extent and produce interferences. I pointed out the bearing of this phenomenon on the theory of the coronas of cloudy condensation (*l. c.*, p. 129), where the compound diffraction spectra, due to successive, parallel, equidistant layers of fog-particles (a sort of space lattice), are superimposed and interfere in a manner evidenced by the disk colors of coronas.

In the actual case of distribution, however, the fog-particles (as I also pointed out) are too far apart to admit of the immediate application of the direct theory in question. Some extension of this point of view must therefore be forthcoming if the experiment with halved gratings one behind the other is to be reconciled with the circumstances under which coronal phenomena appear.

CHAPTER V.

DISPLACEMENT INTERFEROMETRY APPLIED TO THE QUADRANT ELECTROMETER.

67. Apparatus.—In an earlier report experiments were given showing the adaptation of the quadrant electrometer for the measurement of very small potential differences, when the needle is provided with two symmetrical, light, plane mirrors, in parallel. The excursions of the needle may be read off, for small angular deviation, on the displacement interferometer. If $\delta = \Delta N$ is the displacement of the mirror of the micrometer of this instrument, and i the angle of incidence of the ray impinging on either of the small parallel mirrors on the needle,

$$\delta = 2a \cos i \qquad -d\delta/di = 2a \sin i$$

where a is the normal distance apart of the parallel mirrors. If degrees of arc are used the ratio is $0.035\, a \sin i$ and i is usually about 45°. It should be possible with such an arrangement to obtain a sensitiveness of a few millionths volts per vanishing interference ring, and the following paper is a further attempt to reach this result, practically.

The main, if not insuperable, difficulty encountered in such an apparatus is the continual and often irregular drift of the needle, when the condition of rest is so sharply determined. A special environment, without city tremors and at constant temperature, seems to be the only means of obviating these annoyances.

The apparatus used is shown in fig. 63 in vertical section. AA is the perforated base of a massive brass plate, 1 cm. thick, securely fastened by a large clamp to one arm of the interferometer, capable of some rotation around the vertical and horizontal axes for leveling the whole apparatus, etc. To this the quadrants a, b are firmly attached, by aid of screws i, j, but in such a way as to be quite insulated from the brass plate, in view of the perforated columns g, h and nuts u, v of hard rubber and of the form shown. The clamp-screws k, l are in metallic contact with i, j, and carry charges to the quadrants. There are about 2 inches of free space below the plate AA, available for the connections and, if necessary, for a liquid damper, w.

The needle consists of two 8-shaped leaves, c and d (*biplanes*), symmetrically fastened to the stem st, on which the needle is bifilarly suspended from silk fibers. The two small parallel mirrors, e and f, are adjustably attached to a fine metallic wire at right angles to st and in contact with d. Each mirror has a light vertical and horizontal axis in a bit of cork (not shown). The mirrors are first made parallel by using sunlight and then fixed with melted wax, after which the aluminum foils c and d are centered in place, the eyelets at s and t having not as yet been bent. Light reaches the mirrors e, f through two corresponding holes cut in the vertical walls of the quadrants. The weight of such a needle is easily kept within 0.75 gram and the air-damping is quite sufficient. Unfortunately its period is large, being about 1 minute, and it is apt to vibrate as a pendulum. Hence it is often convenient to hook on a wire at *t*, bent like an inverted V, with the free ends submerged in water to secure greater steadiness on the interferometer; or a mica vane may be added, as at *w*.

The bifilar suspension, 13.5 to 20.2 cm. long in the different experiments, terminating above in the hooked brass rod *r*, is adjustably fixed in the brass cylinder *p*, which in turn is secured in the hard-rubber insulator *n*, attached at right angles to the brass standard *GG*, the lower end of which is screwed to the

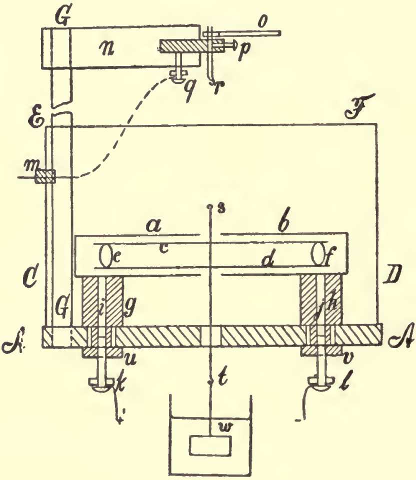


FIG. 63.

brass plate *AA*. This rod can be lengthened telescopically (not shown) admitting of different lengths of bifilar suspension. The hard-rubber lever *O* enables the observer to twist the bifilar. The charge, from a Zamboni cell or the lighting circuit (250 volts), is conveyed to the needle through the hard-rubber insulator at *m* and the clamp-screw at *q* (which in turn secures the plug *p*), through the moistened bifilar wires, as in Dolezalek's apparatus; or it may be admitted through the insulated damper below *t*.

Finally, the lower part of the case *CD* envelops the quadrants more or less permanently and is provided with wide plate-glass windows for observation. The upper part *EF* of the case may be taken off like a hat.

68. Observations.—Experiments were made with this apparatus at considerable length, but they were not sufficiently definite to lead to any quantitative statement. Great difficulty was experienced, in addition to the drift of the needle, in securing an adjustment of the mirrors such that the beam of light might pass from mirror to mirror and return through the inside of the quadrants. For since the mirrors are quite inclosed in the latter, the path of the beam of light can not easily be seen, and it is troublesome to obtain the several reflections to the best advantage. As the needle fits the quadrants with but one-eighth inch of clear space, it is very liable to be unstable if the parts of it are but slightly out of true.

69. Observations, continued.—For this reason it was thought preferable to conduct the experiments by using a needle provided with mirrors attached on the *outside* of the quadrants. Such a needle (No. 1) is shown in fig. 64 (of the wedge type) at *cd*, the two 8-shaped leaves meeting on the outside, in a horizontal circular arc *xy*. The mirrors with the axles in cork are shown at *ef*, and should be several centimeters above the quadrants *ab*. The adjustment here is comparatively easy, as the mirrors and the path of light are all quite visible. The needle, being sharp-edged, may be charged to a potential of several hundred volts, without instability. The period, however, is still large.

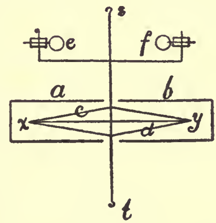


FIG. 64.

In the first series of experiments the needle was charged with a Zamboni cell to about 150 volts, and the voltage measured at the quadrants was about 0.04 volt. The ellipses showed continual drift, the needle moving as if a force acted in one direction for a time large as compared with the period of the needle. The mirrors were slightly curved, so that in place of ellipses the interference figures were lemniscates. In spite of the difficulties, the two series of experiments show sensitiveness of 0.5 and 0.4 cm. per volt, respectively, which is equivalent to about 7×10^{-5} volt per vanishing interference ring.

Using the same needle, the voltage was now presumably doubled by using two Zamboni piles. The sensitiveness, however, not only was not enhanced, but showed a decrease, 0.02 volt being measured. In other experiments the sensitiveness was successively 0.5, 0.4, 0.4 cm. per volt, respectively.

70. Observations, continued.—The sensitiveness was now increased by using a new needle (II) of the form given in figs. 65A and 65B. The two 8-shaped leaves or biplanes, *c* and *d*, of the needle are parallel and the circular edges at *x* and *y* closed with parts of cylindrical shells of aluminum foil. It is presumable from the elementary theory of the instrument that these walls *x*, *y* must contribute essentially to its sensitiveness. In the present case the capsule of the quadrants (II) was about 10 cm. in diameter and about 2 cm. in vertical height, within, with a length of the needle, *xy*, of about 9 cm. and a distance apart of the biplanes *c* and *d* about 0.8 cm. This gave about 0.5 cm. of clear space at the ends and about 0.6 cm. of clear space above and below the needle, as an allowance for stability. The needle swung freely and was inserted without difficulty. The mirrors were about 8 cm. apart. To secure greater steadiness a water damper was installed below, though it would not otherwise have been necessary. The drift was at first marked, but finally subsided to a reasonably small value. The sensitiveness (centimeters of displacement per volt) in successive groups of observations was (ΔV potential increment, ΔN micrometer displacement) $\Delta N/\Delta V = 1.0, 1.1, 1.1, 1.6, 0.8, 1.0, 1.1, 1.2$; therefore about 1.1 cm. per volt, equivalent to 3×10^{-5} volt per vanishing interference ring. In these data some extraneous oscillation of the needle is manifest, which vanishes in consecutive groups of results.

The experiment was then repeated with two Zamboni cells charging the needle. This, however, again actually reduced the sensitiveness to $\Delta N/\Delta V = 0.8, 0.7, 0.6, 0.7, 0.7$ cm. per volt, in successive groups of observations, a result equivalent to 4×10^{-5} volt per vanishing ring. The data remained of the same order, whether the needle was charged from above or below, so that it is inherent in the theory of the instrument. On returning to the single Zamboni charging cell, sensitiveness again increased to $\Delta N/\Delta V = 1.15$, or to about 26×10^{-6} volt per ring.

The Zamboni cells were now removed and the needle charged from above with the electric-lighting circuit at 250 volts. To obviate the effect of drift, which is liable to be persistently in one direction, observations were taken every 1.5 minutes. The sensitiveness in two groups of experiments of about 5 observations each was then 1.1 and 1.4 cm. per volt, or on the average 24×10^{-6} volt per vanishing ring. Many other experiments were made with similar results.

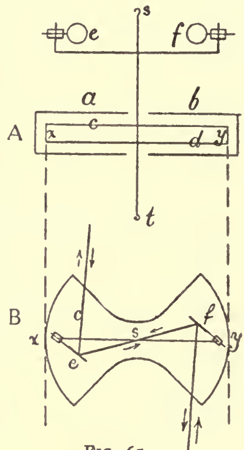


FIG. 65.

The annoyance of a drifting needle, which occurs throughout the above results and which at first seemed to have a definite direction from the dark to the light side of the parallel mirrors, was also made the subject of considerable study, sunlight being used to avoid the radiations from the body of the electric lamp. In these cases the displacement of about 0.07 cm. within a half hour was usually reversed in the course of this time, so as to bring the needle nearly back to its original position. As the experiments were made with the apparatus uncharged, the only reason for this drift seemed therefore to be the occurrence of steady air-currents, in spite of the protection of the case and the rapid subsidence of the pendulum vibrations of the damped needle. The attempts made to obviate these difficulties were all futile.

71. Observations, continued.—Another biplane needle (III) in place of the last (figs. 65A and 65B) was now installed. The blades of the needle were 1.2 cm. apart and to give it stiffness a vertical partition running symmetrically from end to end was fixed within, the whole being of aluminum foil 0.002 cm. thick and the frame, as before, of steel wire 0.044 cm. in diameter. The weight of the needle with mirrors adjusted was about 1.2 grams, the bifilar suspension 22 cm. long and the threads about 0.05 cm. apart. Unfortunately the damping, even with the presence of a water well, proved insufficient and the period too long in the case of the suspension used.

An example of the six groups of results obtained successively, in the measurement of one-fortieth volt, with the needle charged to about 150 volts may be omitted, the mean being about $\Delta N = 0.028$ cm. or $\Delta N/\Delta V = 1.11$ cm. of micrometer displacement per volt, equivalent to 27×10^{-6} volt per ring. In spite of the vertically more extended needle, there has therefore been no advantage in sensitiveness over the preceding case.

The following experiments were made at different voltages, showing better agreement for the larger voltages, in which the drift is less significant, relatively.

ΔV	ΔN	$\Delta N/\Delta V$	Needle.	Quadrant.
volt	*cm.*	*cm.*	*II*	*II*
0.0125	0.0135	1.08		
.0375	.045	1.20		
.0625	.077	1.23		

Further experiments with this needle led to no new results. In particular the endeavor to replace the silk suspension used by a bifilar the threads of which were single fibers of silk, proved a failure owing to the instability of the needle.

72. Further observations.—The same needle was now placed within large quadrants (III), 11 cm. in diameter and 2.3 cm. high within, to obviate the difficulty from instability in a needle carrying 250 volts. While this was accomplished, the drift now became excessively large. The mean results were $\Delta V = 0.0125$ volt; $\Delta N = 0.076$ cm.; $\Delta N/\Delta V = 6.1$ cm.; or about 5×10^{-6} volt per vanishing ring. Unfortunately this large sensitiveness, the largest obtained, could not be controlled.

It appears from these results that in the above cases the actual restoring torque could not have been the torsion of the bifilar, but rather a directed residual electrical attraction between the needle and the quadrants, the torque of the fiber being operative merely in placing the needle in the fiducial position, symmetrically with respect to the division line between the quadrants. In other words, the displacement of the needle is not to be estimated in terms of the rate at which the bifilar torque changes per degree, but in terms of a very much larger coefficient of the electrical forces in question, so that apart from giving position to the needle as specified, the bifilar acts not very differently from a unifilar suspension. The instrument is thus much less sensitive than would be inferred from the dimensions of the bifilar. Hence it appeared desirable to return to the needle and quadrants in § 70, with the object of ascertaining whether the sensitiveness might not be actually increased by decreasing the potential of the needle until a deflection fully corresponding to the torque of the bifilar should show itself. The present point of view also indicates why nothing was gained by the use of a larger needle in § 71, seeing that in such a case the electrical restoring forces increase at the same rate as the deflecting forces arising in the difference of potential of the quadrants. In the same way the negative effect, as regards sensitiveness, of an increase of the potential of the needle above a certain value is accounted for. Table 13 contains data bearing on this inference.

TABLE 13.—Needle II; quadrants II; steel frame.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
	volt	*cm.*	*cm.*
I. 126 volts.....	0.05	0.032	0.64
II. 150 volts.....	.05	.040	.80
III. 183 volts.....	.025	.009	.36
	.05	.025	.50
	.075	.047	.63
	.100	.062	.62

Thus the sensitiveness rapidly reaches a maximum when the potential of the needle is about 150 volts, after which it more gradually diminishes (see fig. 66, curve *b*).

Furthermore, in Series III, where the needle is at the highest potential applied, the sensitiveness seems to increase with the voltage measured. This, however, is merely the result of the fact that there is apparently a small fixed voltaic potential difference between the quadrants, even if they are nominally identical or in the connections. Thus in fig. 66, curve *a*, ΔV and ΔN are pro-

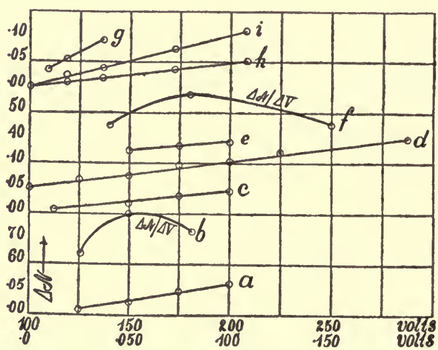


FIG. 66.

portional within the inevitable errors; but the deflections begin with a difference of potential of about 0.012 volt. In measuring such small voltages electrostatically these voltaic differences become of serious moment.

The maximum sensitiveness obtained is not as large as above, being but 40×10^{-6} volt per vanishing ring. Finally, the water damper was removed, so that the needle was subject to air damping only. After a long trial it was necessary to abandon the work, as, in consequence of the excessive drift, measurement was out of the question. Most of this drift is probably introduced by the steel frame.

73. Observations, continued.—The framework of the needles above was of steel. It was supposed that even if not originally magnetic, such a needle might be subject to variations of the earth's field, through which it becomes temporarily magnetic by induction. Accordingly a needle of the same dimensions as the preceding was constructed on a frame of thin copper wire and tested, with the results of table 14.

TABLE 14.—Needle II, copper frame. Quadrants II.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
	volt	*cm.*	*cm.*
140 volts.........	0.0125	0.008	0.64
	.025	.012	.48
	.050	.020	.40
	.075	.035	.47
	.100	.044	.44

The sensitiveness is on the average 70 microvolts per ring, a smaller value than in the last experiments, owing to a somewhat greater weight of the needle. The voltage was now increased further and the following experiments made:

TABLE 14.—Continued.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
180...............	0.025	0.017	0.70
	.050	.025	.50
	.075	.046	.61
	.100	.053	.53
	.125	.071	.57
	.187	.098	.52
250...............	.050	.025	.50
	.075	.033	.44
	.100	.041	.41

As before, the sensitiveness passes through a maximum when the voltage of the needle is about 180, and is about as great for the voltage of 140 volts as for 250 volts (see curve *f*, fig. 66). The maximum sensitiveness is 53 microvolts per ring. Though the drift was not quite removed, the stability of the needle under any given circumstances proved in fact to be greater than before, indicating a marked improvement as the result of replacing the steel frame by one of copper. The curves *c*, *d* (raised 0.05 cm.), *e* (raised 0.1 cm.), show that ΔV and ΔN are proportional within the limits of error. The latter, *e*, seems to begin with an initial potential which would mean that the sensitiveness is even lower at 250 volts than at 140 volts.

It therefore seemed necessary to replace the needle in some other of the above experiments by structures not containing steel. Thus the needle and quadrants used in § 71 with this improvement gave the results shown in table 15.

In view of the low voltage of the needle, the sensitiveness attained is, as above, exceptionally high. The displacement ΔN is proportional to ΔV (see fig. 66, *g*), but begins with a permanent potential of 0.008 volt. Something similar to this occurs in some of the above results on a smaller scale, so that it is not impossible that voltaic potential differences in the quadrants (or connections) may be in question. These were of brass and nominally identical; but a difference of 0.01 volt is not out of the question. Allowing for the initial potential difference, the sensitiveness is about $\Delta N/\Delta V = 2.0$ cm. per volt or 50 microvolts per vanishing interference ring. The needle was far more steady than in the above cases, so that measurements could be made with reasonable assurance. A curious result is thus attained: on widening the quadrants, so that the distance between the quadrants and the needle is increased within limits, greater sensitiveness is secured. The reason has been suggested, that inasmuch as the electric forces which place the needle are now small, the latter is subject to the force of the bifilar suspension only.

TABLE 15.—Large needle III in large quadrants III. Copper frame.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
	volt	*cm.*	
185 volts.........	0.009	0.035	3.8
	.018	.055	3.0
	.036	.093	2.6

Further work was done with the needle at 250 volts; but an adequately stable condition of the needle could not be obtained, as it gradually crept beyond the range of the interferometer.

Finally, experiments were made with a needle of the ordinary form (I, § 69), inclosed in the intermediate quadrants (II). The relatively sharp edges of the needle should reduce the electric torque.

TABLE 16.—Needle I. Quadrants II. Copper frame.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
	volt	*cm.*	
145 volts.........	0.018	0.010	0.56
	.072	.053	.74

The sensitiveness here is not inferior to the usual cases above, being on the average 46 microvolts per ring, and this in conformity with the relatively low potential of the needle. The following results were obtained at higher potentials with the same needle:

TABLE 16.—Continued.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
240.............	0.018	0.011	0.61
	.036	.019	.53
	.072	.035	.50
	.108	.051	.47
	.145	unstable.	

In spite of the much larger potential of the needle in the last series, the average sensitiveness is again less, showing the same behavior as the above cases. The relation of potential and displacement is linear (fig. 66, curve *h*), but the displacements begin with a potential of 0.009 volt. Allowing for this, the sensitiveness is 66 microvolts per vanishing interference ring. The steadiness of the needle in both series of experiments was exceptional for this laboratory, so that the motion of single rings could at times have been counted. Nevertheless the relative values of successive displacements for the same potential increment were not superior to the above.

Since $di = d\delta/2a \sin i$, the electrometer itself in the most sensitive detailed case (needle III, quadrant III) was only moderately sensitive, for if $i = 45°$, $a = 4.5$ cm., $d\delta/\Delta V = \Delta N/\Delta V = 2$ or $\Delta i/\Delta V = 0.32$ radian per volt $= 18.4°$ per volt. Hence, the reflected ray in the ordinary mirror and scale adjustment, at 1 meter distance of scale from mirror, would move over about 64 cm. per volt. In one of the incidental cases above, it is true, about three times this value was reached. In the other cases it was proportionately less sensitive. Thus for $\Delta N/\Delta V = 0.5$, the deflection would be but 16 cm. per volt. No doubt much could be accomplished by making the electrometer itself more sensitive; but this improvement was not the immediate purpose of the present article.

Other comparative experiments with copper-framed needles were now made. The sharp-edged needle, I, placed in the large quadrants III gave the results of table 17.

TABLE 17.—Needle I. Quadrants III. Copper frame.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
	volt	*cm.*	
250	0.018	0.010	0.56
	.036	.016	.45
	.072	.040	.56
	.108	.081	.75
	.144	.115	.80

In this case the displacements are not proportional to the voltages, but increase at an accelerated rate. Neither do they seem to begin at the origin. The sensitiveness accordingly increases rapidly with increased deflection, but its mean value is of the ordinary magnitude. This behavior of a thin needle, in relatively wide quadrants, where stability should have been insured, was unexpected; but by raising the needle the following results were found, showing that the needle above was inadequately centered:

TABLE 17.—Continued.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
250	0.018	0.023	1.3
	.036	.038	1.06
	.072	.075	1.04
	.108	.111	1.03

This result is a great improvement; for not only is the potential proportional to the displacement (see curve *i*, fig. 66), but the sensitiveness is much larger than heretofore, for the same needle, being about 30 microvolts per vanishing ring. The advantage of wide quadrants is thus again sustained. The sensitiveness, however, lags behind the result for the large biplane needle III (curve *g*) under the same circumstances.

The intermediate biplane needle II in the quadrants III shows the results of table 18.

TABLE 18.—Needle II. Quadrants III. Copper frame.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
	volt	*cm.*	
250 volts.........	0.018	0.007	0.39
	.036	.022	.61
	.072	.050	.70
	.108	.068	.63
	.144	.095	.66

The sensitiveness is low, owing again, no doubt, to the position of the needle. Otherwise the observations are good. The needle was then raised, with the following results:

TABLE 18.—Continued.

Needle at—	ΔV	ΔN	$\Delta N/\Delta V$
250...............	0.036	0.023	0.64
	.072	.050	.69
	.108	.078	.72
	.144	.103	.72
	.018	.015	.83

The sensitiveness has been slightly increased. Moreover, it grows larger in the course of the work, as if some surface viscosity in the liquid of the damper were gradually overcome. The instrument in general behaved admirably, barring alone the presence of drift which can not in the present laboratory be quite overcome. As the fibers were but 13.5 cm. long as compared with 23.0 cm. above, the mean reduced sensitiveness was 25 microvolts per ring. It is thus inferior both to the large biplane and to the wedge, for reasons which do not appear. The sensitiveness should have been intermediate.

Finally, the intermediate quadrants II were again mounted with the same needle, the quadrants being specially *smoothed* inside, so as possibly to eliminate electric restoring forces. The results, however, were not essentially different from the above.

74. Summary.—The results of this long and excessively laborious paper may be given in a few words. By providing the needle of the quadrant electrometer with a pair of mirrors, in parallel, and observing displacements on the interferometer, voltages as small as 10 microvolts may be detected per vanishing interference ring, so that a single microvolt should be reached by estimation. In the above experiments this could not be done, because the needle was never confined to a fixed position of equilibrium, to an extent compatible with the use of light-waves. The causes of this drift are incidental, probably attributable to air-currents, convection currents due to temperature differences and pendulum motion of the needle resulting from tremors. Steel must always be excluded from the framework of the needle.

The sensitiveness as is otherwise known, theoretically, does not in any case increase with the potential of the needle, but passes through a maximum (in the above designs) usually at about 150 volts. This is the case both with the sharp-edged and the cylindrically-faced biplane needles. The directing force in the case of such needles is essentially electric; *i.e.*, they are set in a position of equilibrium relatively to the quadrants by electric stress large in comparison with the torque of the bifilar. As soon as these forces increase at the same rate as the potential of the needle, the further increase of the latter is no longer serviceable. Hence the biplane needle, set in relatively wide quadrants, was found to offer the best conditions of sensitiveness, and it is in the case of needles and quadrants of this design that the best results were obtained. In other words, the sensitiveness also passes through a maximum as the mean distance between the outside contours of the needle and the inside contours of the quadrants increases.

After preliminary experiments, the optics of the instrument offered no serious difficulty. It is merely necessary to follow the reflected light by placing white screens behind each mirror in the direction of the impinging rays. Since the rays are reflected at the grating, the returning ray also necessarily passes through the grating, and this part of the adjustment is therefore automatic. With a copper-framed needle, the water damper will probably not be essential, in which case the discrepancies due to surface viscosity will also disappear.